Computing Tamagawa numbers of hyperelliptic curves

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Question

How can $c_{X/K}$ be computed efficiently?
Can one find formulae for Tamagawa numbers in (possibly degenerating) families?

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1. find the dual graph G of X/K, along with its metric and its Frobenius action;
2. from the dual graph G, calculate the Tamagawa number:
    - let $\Lambda = \mathrm{H}_1(G)$ be the (integral) homology lattice;
    - consider the embedding $\Lambda \hookrightarrow \Lambda^\vee$ induced by the intersection-length pairing;
    - $\Lambda^\vee/\Lambda$ is the group of components of the Néron model of $\mathrm{Jac}(X)/K^{\mathrm{nr}}$;
    - $(\Lambda^\vee/\Lambda)^{\mathrm{Fr}}$ is the group of components of the Néron model of $\mathrm{Jac}(X)/K$;
    - $c_{X/K} = \#(\Lambda^\vee/\Lambda)^{\mathrm{Fr}}$.

The hyperelliptic algorithm

BY trees

Theorem (Dokchitser–Dokchitser–Maistret–Morgan)

Let X/K be a hyperelliptic curve with dual graph G, and let ι denote the hyperelliptic involution. Then $T = G/\iota$ is a tree.

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Moreover, G can be reconstructed up to homeomorphism from the pair (T, S), where $S \subseteq T$ is the ramification locus of $G \to T$. We will call such a pair (T, S) a *BY tree**.

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We will formulate a precise and efficient version of the previous algorithm for hyperelliptic curves, replacing the dual graph with its corresponding BY tree.

Overview of the algorithm

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Second step: From the BY tree, compute $c_{X/K}$ (purely graph-theoretically). [B.]

Step 1: BY trees from explicit equations

Given an explicit equation $y^2 = f(x)$ for a hyperelliptic curve X/K, there is a naturally associated *cluster picture*, namely the picture formed by drawing the set of roots of f in $\overline{K}$, and drawing circles around all the subsets of $\mathrm{Root}_{\overline{K}}(f)$ cut out by discs in $\overline{K}$.

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- let $S_0 \subseteq T_0$ be the subgraph whose edges are the edges as above where the child cluster has odd size, and the vertices of S_0 are just the endpoints of all such edges;
- finally, produce (T, S) from (T_0, S_0) by deleting all the vertices of T_0 (also S_0) corresponding to clusters of size 1.

Example

Consider the hyperelliptic curve $X/\mathbb{Q}_3$, given by the equation

$$y^2 = ((x-i)^2 - 3^a)((x+i)^2 - 3^a)(x^2 - 3^b)((x-1)^2 - 3^c).$$

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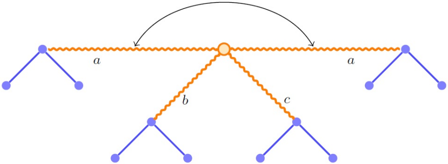

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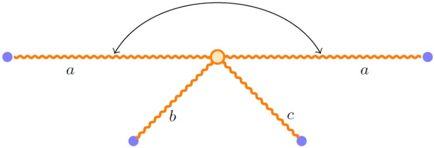

Step 2: Tamagawa numbers from BY trees

Definition

If $\mathcal{T} = (T, S)$ is a BY tree (with metric & Frobenius), we let $\Lambda = \mathrm{H}_1(T, S)$ be the relative homology lattice, and $\Lambda \hookrightarrow \Lambda^\vee$ the embedding induced by the intersection-length pairing. The quantity

$$c_{\mathcal{T}} := \#(\Lambda^\vee/\Lambda)^{\mathrm{Fr}}$$

is called the *Tamagawa number* of $\mathcal{T}$.

When $\mathcal{T}$ is the BY tree associated to a hyperelliptic curve X/K, $c_{\mathcal{T}} = c_{X/K}$ calculates the Tamagawa number of X.

Step 2: Tamagawa numbers from BY trees

Definition

If $T = (T, S)$ is a BY tree (with metric & Frobenius), we let $\Lambda = \mathrm{H}_1(T, S)$ be the relative homology lattice, and $\Lambda \hookrightarrow \Lambda^\vee$ the embedding induced by the intersection-length pairing. The quantity

$$c_T := \#(\Lambda^\vee/\Lambda)^{\mathrm{Fr}}$$

is called the *Tamagawa number* of T.

When T is the BY tree associated to a hyperelliptic curve X/K, $c_T = c_{X/K}$ calculates the Tamagawa number of X.

We want to find a graph-theoretic method for calculating c_T.

Reduction to simple BY trees

For a general BY tree (T, S), $T \setminus S$ may have many components. The closure of a component is itself a BY tree, and the Tamagawa number of (T, S) is the product of the Tamagawa numbers of some of these components, one for each Fr-orbit in $\pi_0(T \setminus S)$.

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In this way, we can reduce the calculation of Tamagawa numbers to the calculation for *simple* BY trees – those for which S is a subset of the leaves of T. These come in two types, according to the sign of Frobenius.

Tamagawa numbers of (positive) simple BY trees

Formula (B.)

Let (T, S) be a positive simple BY tree with metric and Frobenius. Write $\overline{T} = T/\mathrm{Fr}$ for the quotient tree, and give $\overline{T}$ the metric where an edge $\overline{e}$ corresponding to a Fr-orbit of q edges of length l is given length $l(\overline{e}) = l/q$. Write $\overline{S} = S/\mathrm{Fr} \subseteq \overline{T}$, and Q for the product of the sizes of the Fr-orbits in S.

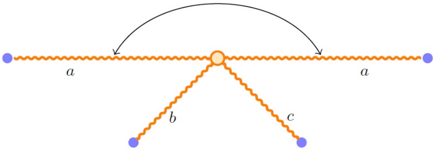
Then

$$c_T = Q \sum \prod_{r=1}^{|\overline{S}|-1} l(\overline{e}_1) l(\overline{e}_2) \dots l(\overline{e}_{|\overline{S}|-1}),$$

where the sum is taken over all unordered $(|\overline{S}| - 1)$-tuples of edges of $\overline{T}$ which disconnect the points of $\overline{S}$ from one another.

Example

Continuing the earlier example, let's calculate the Tamagawa number of the following BY tree:



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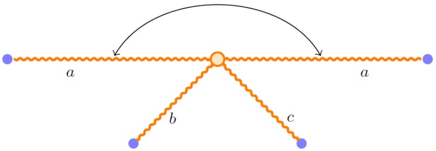
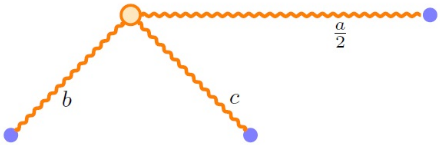


It is positive and simple, so the formula on the previous slide applies. The product of the sizes of the Frobenius orbits on S is $Q = 2$.

Example (cont.)

The quotient tree $\overline{T}$ is



and the removal of any two edges disconnects the points of $\overline{S}$, so that the Tamagawa number of T is

$$c_T = 2\left(\frac{a}{2}b + \frac{a}{2}c + bc\right) = ab + ac + 2bc.$$

Any questions?