Chabauty–Kim and the Section Conjecture for locally geometric sections

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The Selmer Section Conjecture

Anabelian geometry

- $Y/\mathbb{Q}$ is a smooth projective curve of genus ≥ 2.
- Fundamental exact sequence

$$1 \to \pi_1^{\mathrm{et}}(Y_{\overline{\mathbb{Q}}}) \to \pi_1^{\mathrm{et}}(Y) \to G_{\mathbb{Q}} \to 1 \qquad \text{(FES)}$$

- A rational point $y \in Y(\mathbb{Q})$ induces a splitting of (FES), well-defined up to $\pi_1^{\mathrm{et}}(Y_{\overline{\mathbb{Q}}})$-conjugation. We get a *section map*

$$s\colon Y(\mathbb{Q}) \to \mathrm{Sec}(Y/\mathbb{Q}) := \{\text{splittings of (FES)}\}/\sim \ .$$

Section Conjecture (Grothendieck, 1983)

If $Y/\mathbb{Q}$ is a smooth projective curve of genus ≥ 2, then s is bijective.

The Section Conjecture

Section Conjecture (Grothendieck, 1983)

If $Y/\mathbb{Q}$ is a smooth projective curve of genus ≥ 2, then s is bijective.

- s is known to be injective. Surjectivity is unknown.
- Particular examples of Y have been constructed where $\mathrm{Sec}(Y/\mathbb{Q}) = \emptyset$ (Stix, Li–Litt–Salter–Srinivasan). In these cases, s is bijective automatically.
- In general, $\mathrm{Sec}(Y/\mathbb{Q})$ is very mysterious (we don't know $G_{\mathbb{Q}}$!).

Motivating questions

Is the set $\mathrm{Sec}(Y/\mathbb{Q})$ finite? Can we compute it in particular examples?

Selmer sections

For $s \in \mathrm{Sec}(Y/\mathbb{Q})$ a section, the restriction $s|_{G_p}$ is a section of the local fundamental exact sequence

$$1 \to \pi_1^{\mathrm{et}}(Y_{\overline{\mathbb{Q}}_p}) \to \pi_1^{\mathrm{et}}(Y_{\mathbb{Q}_p}) \to G_p \to 1\,.$$

Definition (Selmer section set)

We say that s is *Selmer* (or sometimes *locally geometric* or *adelic*) just when $s|_{G_p}$ comes from a $\mathbb{Q}_p$-rational point $y_p \in Y(\mathbb{Q}_p)$ for all primes p, and write $\mathrm{Sec}(Y/\mathbb{Q})^{\mathrm{Sel}}$ for the set of Selmer sections. So we have

$$Y(\mathbb{Q}) \subseteq \mathrm{Sec}(Y/\mathbb{Q})^{\mathrm{Sel}} \subseteq \mathrm{Sec}(Y/\mathbb{Q})\,.$$

Conjecture (Selmer Section Conjecture)

If $Y/\mathbb{Q}$ is a smooth projective curve of genus ≥ 2, then

$$\mathrm{Sec}(Y/\mathbb{Q})^{\mathrm{Sel}} = Y(\mathbb{Q})\,.$$

Main point of this talk:

> *One can use p-adic methods from the study of rational points to study the Selmer section set.*

Two instances of this idea:

- Using the Lawrence–Venkatesh method to prove Mordell-like finiteness theorems for the Selmer section set (B.–Stix 2022).
- Using the Chabauty–Kim method to sometimes compute the Selmer section set (B.–Kumpitsch–Lüdtke 2023).

Theorem A

If s is a Selmer section and p is prime, then the local point $y_p \in Y(\mathbb{Q}_p)$ such that $s|_{G_p} = s(y_p)$ is unique. The assignment $s \mapsto y_p$ is a function, the *localisation map* at p

$$\mathrm{loc}_p \colon \mathrm{Sec}(Y/\mathbb{Q})^{\mathrm{Sel}} \to Y(\mathbb{Q}_p)\,.$$

Theorem A (B.–Stix 2022)

For any smooth projective curve $Y/\mathbb{Q}$ of genus ≥ 2, the image of

$$\mathrm{loc}_p \colon \mathrm{Sec}(Y/\mathbb{Q})^{\mathrm{Sel}} \to Y(\mathbb{Q}_p)$$

is finite for every prime p.

S-Selmer sections

Let S be a finite set of primes. If $Y/\mathbb{Q}$ is a smooth hyperbolic curve and $\mathcal{Y}/\mathbb{Z}_S$ is an S-integral model, we say that a section $s \in \mathrm{Sec}(Y/\mathbb{Q})$ is *S-Selmer* (with respect to $\mathcal{Y}$) just when $s|_{G_p}$ comes from a

$$\begin{cases} \mathbb{Z}_p\text{-integral point } y_p \in \mathcal{Y}(\mathbb{Z}_p) & \text{for } p \notin S \\ \mathbb{Q}_p\text{-rational point } y_p \in Y(\mathbb{Q}_p) & \text{for } p \in S. \end{cases}$$

We write

$$\mathrm{Sec}(\mathcal{Y}/\mathbb{Z}_S)^{\mathrm{Sel}} \subseteq \mathrm{Sec}(Y/\mathbb{Q})$$

for the set of S-Selmer sections. This contains the set $\mathcal{Y}(\mathbb{Z}_S)$ of S-integral points on $\mathcal{Y}$.

Conjecture (S-Selmer Section Conjecture)

$$\mathrm{Sec}(\mathcal{Y}/\mathbb{Z}_S)^{\mathrm{Sel}} = \mathcal{Y}(\mathbb{Z}_S)\,.$$

Theorem B

Theorem B (B.–Kumpitsch–Lüdtke 2023)

Let $S = \{2\}$ and let $\mathcal{Y} = \mathbb{P}^1_{\mathbb{Z}[1/2]} \smallsetminus \{0, 1, \infty\}$ be the thrice-punctured line over $\mathbb{Z}[1/2]$. Then

$$\mathrm{Sec}(\mathcal{Y}/\mathbb{Z}[1/2])^{\mathrm{Sel}} = \mathcal{Y}(\mathbb{Z}[1/2]) = \{2, -1, 1/2\}\,.$$

This result was already known (for any S) by Jakob Stix. Our method of proof is different, and is more computational in nature.

p-adic obstructions

General setup

- $Y/\mathbb{Q}$ is a smooth projective curve of genus ≥ 2.
- $\mathbb{E}$ is a $\mathbb{Q}_p$-local system on Y_{et} (for p a fixed prime).
- For every $y \in Y(\mathbb{Q})$, the fibre $\mathbb{E}_y$ is a $G_{\mathbb{Q}}$-representation.
- For every $y_\ell \in Y(\mathbb{Q}_\ell)$, the fibre $\mathbb{E}_{y_\ell}$ is a G_ℓ-representation.
- Fix local conditions: for every prime ℓ (including p), fix a set $\mathfrak{R}_\ell$ of isomorphism classes of G_ℓ-representations, containing all $[\mathbb{E}_{y_\ell}]$.

Definition (Naïve obstruction locus)

We define

$$Y(\mathbb{Q}_p)_{\mathbb{E}} \subseteq Y(\mathbb{Q}_p)$$

to be the set of points $y_p \in Y(\mathbb{Q}_p)$ such that $\mathbb{E}_{y_p}$ is the restriction to G_p of a $G_{\mathbb{Q}}$-representation V such that $[V|_{G_\ell}] \in \mathfrak{R}_\ell$ for all primes ℓ. So

$$Y(\mathbb{Q}) \subseteq Y(\mathbb{Q}_p)_{\mathbb{E}} \subseteq Y(\mathbb{Q}_p)\,.$$

Abelian-by-finite families

It is often advantageous to work with structured $\mathbb{Q}_p$-local systems.

Definition (Abelian-by-finite family)

An *abelian-by-finite family*

$$X \to Y' \to Y$$

over Y consists of a finite étale covering $Y' \to Y$ and a polarised abelian scheme $X \to Y'$.

- The relative étale cohomology $\mathbb{E}^i := \mathrm{R}^i\pi_*\mathbb{Q}_{p,X}$ is a $\mathbb{Q}_p$-local system. Its fibre at y is $\mathrm{H}^i_{\mathrm{et}}(X_{y,\overline{\mathbb{Q}}}, \mathbb{Q}_p)$.
- $\mathbb{E}^0$ has the structure of an algebra.
- $\mathbb{E}^1$ has the structure of a symplectic $\mathbb{E}^0$-module.

The Lawrence–Venkatesh locus

Definition (Lawrence–Venkatesh locus)

Let $X \to Y' \to Y$ be an abelian-by-finite family, and let p be prime. We define the *Lawrence–Venkatesh locus*

$$Y(\mathbb{Q}_p)^{\mathrm{LV}}_X \subseteq Y(\mathbb{Q}_p)$$

to be the set of points $y_p \in Y(\mathbb{Q}_p)$ such that $\mathbb{E}^0_{y_p} = \mathrm{H}^0_{\mathrm{et}}(X_{y_p,\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ and $\mathbb{E}^1_{y_p} = \mathrm{H}^1_{\mathrm{et}}(X_{y_p,\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ are the restrictions to G_p of an algebra A and symplectic A-module V in the category of $G_{\mathbb{Q}}$-representations satisfying certain local conditions (which we omit). We have

$$Y(\mathbb{Q}) \subseteq Y(\mathbb{Q}_p)^{\mathrm{LV}}_X \subseteq Y(\mathbb{Q}_p)\,.$$

The Chabauty–Kim locus

- Fix $y_0 \in Y(\mathbb{Q})$ and let $\mathbb{E}_\infty$ be the universal pointed geometrically pro-unipotent $\mathbb{Q}_p$-local system on Y_{et}.
- $\mathbb{E}_\infty$ has the structure of a $\mathcal{O}(U_\infty)^\vee$-module-coalgebra in the pro-category of $\mathbb{Q}_p$-local systems, where U_∞ is the $\mathbb{Q}_p$-pro-unipotent étale fundamental group.

Definition (Chabauty–Kim locus)

Let p be a prime of good reduction. We define the *Chabauty–Kim locus*

$$Y(\mathbb{Q}_p)^{\mathrm{CK}}_\infty \subseteq Y(\mathbb{Q}_p)$$

to be the set of points $y_p \in Y(\mathbb{Q}_p)$ such that $\mathbb{E}_{\infty,y_p}$ is the restriction to G_p of a $\mathcal{O}(U_\infty)^\vee$-module-coalgebra in the pro-category of $G_\mathbb{Q}$-representations satisfying certain local conditions (which we omit). We have

$$Y(\mathbb{Q}) \subseteq Y(\mathbb{Q}_p)^{\mathrm{CK}}_\infty \subseteq Y(\mathbb{Q}_p)\,.$$

p-adic obstructions constrain Selmer sections

Recall that we have the localisation map

$$\mathrm{loc}_p \colon \mathrm{Sec}(Y/\mathbb{Q})^{\mathrm{Sel}} \to Y(\mathbb{Q}_p)$$

on Selmer sections.

Lemma ("L–V obstruction constrains Selmer sections")

For all abelian-by-finite families $X \to Y' \to Y$, we have

$$\mathrm{im}(\mathrm{loc}_p) \subseteq Y(\mathbb{Q}_p)^{\mathrm{LV}}_X .$$

Lemma ("C–K obstruction constrains Selmer sections")

If p is a prime of good reduction, then we have

$$\mathrm{im}(\mathrm{loc}_p) \subseteq Y(\mathbb{Q}_p)^{\mathrm{CK}}_\infty .$$

Idea of proof

Lemma ("Naïve obstruction constrains Selmer sections")

Let $\mathbb{E}$ be a $\mathbb{Q}_p$-local system on Y_{et}, and let $(\mathfrak{R}_\ell)_\ell$ be local conditions. Then we have

$$\mathrm{im}(\mathrm{loc}_p) \subseteq Y(\mathbb{Q}_p)_{\mathbb{E}}\,.$$

Proof: $\mathbb{E}$ corresponds to a continuous representation V of $\pi_1^{\mathrm{et}}(Y)$. Given a section $s\colon G_{\mathbb{Q}} \to \pi_1^{\mathrm{et}}(Y)$ of (FES), we can restrict the $\pi_1^{\mathrm{et}}(Y)$-action on V along s to obtain a $G_{\mathbb{Q}}$-representation V_s. If s is Selmer with $\mathrm{loc}_\ell(s) = y_\ell$ then

$$V_s|_{G_\ell} \cong \mathbb{E}_{y_\ell}\,.$$

So $[V_s|_{G_\ell}] \in \mathfrak{R}_\ell$ for all ℓ. Since $\mathbb{E}_{y_p} = V_s|_{G_p}$, we have $y_p \in Y(\mathbb{Q}_p)_{\mathbb{E}}$. □

Proof of Theorem A

Theorem A, recap

Theorem A (B.–Stix 2022)

For any smooth projective curve $Y/\mathbb{Q}$ of genus ≥ 2, the image of

$$\mathrm{loc}_p \colon \mathrm{Sec}(Y/\mathbb{Q})^{\mathrm{Sel}} \to Y(\mathbb{Q}_p)$$

is finite for every prime p.

Follows from $\mathrm{im}(\mathrm{loc}_p) \subseteq Y(\mathbb{Q}_p)^{\mathrm{LV}}_X$ and

Theorem (Lawrence–Venkatesh, B.–Stix)

For every smooth projective curve $Y/\mathbb{Q}$ of genus ≥ 2 and every prime p, there exists an abelian-by-finite family $X \to Y' \to Y$ such that the Lawrence–Venkatesh locus

$$Y(\mathbb{Q}_p)^{\mathrm{LV}}_X$$

is finite.

Proof of Theorem B

Some Conjectures

Let $Y/\mathbb{Q}$ be a smooth hyperbolic curve, $\mathcal{Y}/\mathbb{Z}_S$ a model.

Conjecture (S-Selmer Section Conjecture)

$$\mathrm{Sec}(\mathcal{Y}/\mathbb{Z}_S)^{\mathrm{Sel}} = \mathcal{Y}(\mathbb{Z}_S)\,.$$

Let $\mathcal{Y}(\mathbb{A}_S)_\bullet^{\text{f-cov}} \subseteq \mathcal{Y}(\mathbb{A}_S)_\bullet$ be the finite descent locus.

Conjecture ($\approx$ Stoll's Conjecture)

$$\mathcal{Y}(\mathbb{A}_S)_\bullet^{\text{f-cov}} = \mathcal{Y}(\mathbb{Z}_S)\,.$$

Assume $Y(\mathbb{Q}) \neq \emptyset$. For $p \notin S$ of good reduction, let $\mathcal{Y}(\mathbb{Z}_p)^{\mathrm{CK}}_{S,\infty} \subseteq \mathcal{Y}(\mathbb{Z}_p)$ be the S-integral Chabauty–Kim locus.

Conjecture (Kim's Conjecture)

$$\mathcal{Y}(\mathbb{Z}_p)^{\mathrm{CK}}_{S,\infty} = \mathcal{Y}(\mathbb{Z}_S)\,.$$

Relating the conjectures, I

Theorem (folklore?)

$\mathcal{Y}$ satisfies the S-Selmer Section Conjecture if and only if it satisfies Stoll's Conjecture, i.e.

$$\mathrm{Sec}(\mathcal{Y}/\mathbb{Z}_S)^{\mathrm{Sel}} = \mathcal{Y}(\mathbb{Z}_S) \quad \textit{iff} \quad \mathcal{Y}(\mathbb{A}_S)_\bullet^{\text{f-cov}} = \mathcal{Y}(\mathbb{Z}_S)\,.$$

One direction (S-Selmer⇒Stoll) follows from a result of Harari–Stix.

Theorem (Harari–Stix 2012)

The image of the total localisation map

$$\mathrm{loc}\colon \mathrm{Sec}(\mathcal{Y}/\mathbb{Z}_S)^{\mathrm{Sel}} \to \mathcal{Y}(\mathbb{A}_S)_\bullet$$

is exactly the finite descent locus $\mathcal{Y}(\mathbb{A}_S)_\bullet^{\text{f-cov}}$.

The converse direction is a little more delicate, since loc is not known to be injective. It uses in an essential way a result of Stoll.

Theorem (Theorem of the Diagonal, Stoll 2007 $+\epsilon$)

Let $Z \subset Y$ *be a finite subscheme. If* $y = (y_p)_p \in \mathcal{Y}(\mathbb{A}_S)_\bullet^{\text{f-cov}}$ *satisfies* $y_p \in Z(\mathbb{Q}_p)$ *for a density* 1 *set of primes* p*, then* $y \in \mathcal{Y}(\mathbb{A}_S)$.

Relating the conjectures, II

Theorem (B.–Kumpitsch–Lüdtke)

If $(\mathcal{Y}, S)$ satisfies Kim's Conjecture for all primes p in a set $\mathfrak{P}$ of Dirichlet density 1, then it satisfies the S-Selmer Section Conjecture.

Proof outline:

1. Let $y = (y_p)_p \in \mathcal{Y}(\mathbb{A}_S)^{\text{f-cov}}_{\bullet}$.
2. Harari–Stix $\Rightarrow$ $y = \mathrm{loc}(s)$ for an S-Selmer section s.
3. $y_p = \mathrm{loc}_p(s) \in \mathcal{Y}(\mathbb{Z}_p)^{\mathrm{CK}}_{S,\infty} = \mathcal{Y}(\mathbb{Z}_S)$ for all $p \in \mathfrak{P}$, by Kim's Conjecture.
4. Theorem of the Diagonal $\Rightarrow$ $y \in \mathcal{Y}(\mathbb{Z}_S)$. So we've shown Stoll's Conjecture for $(\mathcal{Y}, S)$. □

The main calculation

This gives a strategy for proving instances of the S-Selmer Section Conjecture: prove Kim's Conjecture for 100% of primes p. We will prove

Theorem B' (B.–Kumpitsch–Lüdtke)

Let $\mathcal{Y} = \mathbb{P}^1_{\mathbb{Z}[1/2]} \smallsetminus \{0, 1, \infty\}$ be the thrice-punctured line over $\mathbb{Z}[1/2]$. Then Kim's Conjecture holds for $(\mathcal{Y}, \{2\}, p)$ for all odd primes p. That is,

$$\mathcal{Y}(\mathbb{Z}_p)^{\mathrm{CK}}_{\{2\},\infty} = \mathcal{Y}(\mathbb{Z}[1/2]) = \{2, -1, 1/2\}\,.$$

In particular, the S-Selmer Section Conjecture and Stoll's Conjecture holds for $(\mathcal{Y}, \{2\})$.

Local conditions for Chabauty–Kim

In the literature, there are several different choices made for the local conditions $\mathfrak{R}_\ell$. In Kim's original formulation, one would take

$$\mathfrak{R}_\ell = \{\text{unramified reps}\} \quad \text{for } \ell \notin \{2, p\}, \text{ and } \quad \mathfrak{R}_p = \{\text{crystalline reps}\}$$

and no condition at 2 ($\mathfrak{R}_2 = \{\text{all reps}\}$). We denote the original Chabauty–Kim locus defined using these local conditions by $\mathcal{Y}(\mathbb{Z}_p)^{\mathrm{CK}_0}_{\{2\},\infty}$.

The version of Chabauty–Kim we are using imposes a non-trivial condition at 2 ("refined Chabauty–Kim", B.–Dogra 2020). The relationship with the original locus is

$$\mathcal{Y}(\mathbb{Z}_p)^{\mathrm{CK}}_{\{2\},\infty} \subseteq \mathcal{Y}(\mathbb{Z}_p)^{\mathrm{CK}_0}_{\{2\},\infty}.$$

Theorem (Dan-Cohen–Wewers 2013)

The original Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)^{\mathrm{CK}_0}_{\{2\},\infty}$ is contained in the common vanishing locus of the functions

$$2\,\mathrm{Li}_2^p(z) - \log^p(z)\,\mathrm{Li}_1^p(z) \quad \text{and}$$

$$24\log^p(2)\zeta^p(3)\,\mathrm{Li}_4^p(z) + \frac{8}{7}\left(\log^p(2)^4 + 24\,\mathrm{Li}_4^p(1/2)\right)\log^p(z)\,\mathrm{Li}_3^p(z)$$

$$-\left(\frac{4}{21}\log(2)^4 + \frac{32}{7}\,\mathrm{Li}_4^p(1/2) + \log^p(2)\zeta^p(3)\right)\log^p(z)^3\,\mathrm{Li}_1^p(z)\,.$$

Proposition (B.–Kumpitsch–Lüdtke)

The refined Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)^{\mathrm{CK}}_{\{2\},\infty}$ is contained in the union of the S_3-translates of the common vanishing locus of the functions

$$\log^p(z) \quad \text{and} \quad \mathrm{Li}_k^p(z) \quad \text{for } k \geq 2 \text{ even.}$$

A computation

Lemma

The only common zero of the functions

$$\log^p(z) \quad \text{and} \quad \mathrm{Li}_k^p(z) \quad \text{for } k \geq 2 \text{ even}$$

on $\mathcal{Y}(\mathbb{Z}_p)$ is $z = -1$.

Remark: We believe that the only common zero of the functions $\log^p(z)$ and $\mathrm{Li}_2^p(z)$ should be $z = -1$. We have verified this computationally for $3 \leq p \leq 10^5$, but don't know how to prove this in general.

Suppose that $z = \zeta$ is a zero of $\log^p(z)$ and $\mathrm{Li}_k^p(z)$ for all even $k \geq 2$. Then ζ must be a root of unity, with $\zeta^p = \zeta$. So ζ is a zero of the modified p-adic polylogarithm

$$\mathrm{Li}_k^{(p)}(z) := \mathrm{Li}_k^p(z) - p^{-k}\,\mathrm{Li}_k^p(z^p)\,.$$

The modified p-adic polylogarithm is congruent modulo p to $(1-z^p)^{-1}\,\mathrm{li}_k^p(z)$ where

$$\mathrm{li}_k^p(z) := \sum_{j=1}^{p-1} j^{-k} z^j \in \mathbb{F}_p[z]\,.$$

So the reduction $\bar{\zeta}$ of ζ is a zero of $\mathrm{li}_k^p(z)$ for all even $k \geq 2$. In particular, $\bar{\zeta}$ is a zero of

$$\mathrm{li}_{p-3}^p(z) = \sum_{j=1}^{p-1} j^{3-p} z^j = \sum_{j=1}^{p-1} j^2 z^j = z(z+1)(z-1)^{p-3}\,.$$

The only zero of this function in $\mathcal{Y}(\mathbb{F}_p)$ is $z = -1$, and hence $\zeta = -1$. $\square$

Assembling the proof

1. The only common zero of $\log^p$ and Li_k^p for $k \geq 2$ even is $z = -1$.
2. The explicit description of the refined Chabauty–Kim locus gives

$$\mathcal{Y}(\mathbb{Z}_p)^{\mathrm{CK}}_{\{2\},\infty} \subseteq S_3\text{-orbit of } -1 = \{2, -1, 1/2\},$$

   so Kim's Conjecture holds.
3. This implies that the S-Selmer Section Conjecture holds for $(\mathcal{Y}, \{2\})$. □

Summary

Summary

One can use p-adic obstructions to study the Selmer section set.

- p-adic obstructions cut out loci inside $Y(\mathbb{Q}_p)$ containing the image of the localisation map

$$\mathrm{loc}_p \colon \mathrm{Sec}(Y/\mathbb{Q})^{\mathrm{Sel}} \to Y(\mathbb{Q}_p)\,.$$

- By controlling these loci, we control the localisation image.
- The Lawrence–Venkatesh obstruction proves finiteness of all localisation images.
- The Chabauty–Kim obstruction can often be used to show that the localisation image is exactly $Y(\mathbb{Q})$. Doing this for 100% of primes p proves the Selmer Section Conjecture for Y.

Thank you for listening

Number fields

Definition

A finite place v of a number field K is called *self-conjugate* just when its restriction to any CM subfield of K is invariant under conjugation.

- If K contains no CM subfield, then every place is self-conjugate.
- A self-conjugate place which is unramified over $\mathbb{Q}$ is called *friendly* in Lawrence–Venkatesh.

Theorem (B.–Stix 2022)

Let K be a number field and Y/K a smooth projective curve of genus ≥ 2. Then the image of

$$\mathrm{loc}_v : \mathrm{Sec}(Y/K)^{\mathrm{Sel}} \to Y(K_v)$$

is finite for every self-conjugate place v of K.