Galois sections and the Lawrence–Venkatesh method

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Introduction

Galois sections

Let K be a number field and Y/K a connected smooth proper curve of genus ≥ 2. The structure map $Y \to \mathrm{Spec}(K)$ induces a map

$$\pi_1^{\text{ét}}(Y) \to G_K \qquad (*)$$

on profinite étale fundamental groups.

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Technical point: One officially has to make choices of basepoint to make sense of the above Galois and fundamental groups. All the maps between profinite groups should be taken to denote *outer* homomorphisms.

The section conjecture

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- The map $Y(K) \to \mathrm{Sec}(Y/K)$ is known to be injective.
- Surjectivity is only known in a handful of cases where $\mathrm{Sec}(Y/K)$ can be shown to be empty.
- Grothendieck believed that a proof of the Section Conjecture would lead to a "topological" proof of the Mordell Conjecture.

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Consequence of the Section Conjecture + Faltings' Theorem

The set $\mathrm{Sec}(Y/K)$ is finite.

Locally geometric sections

If v is a place of K, then the structure map $Y_{K_v} \to \mathrm{Spec}(K_v)$ induces a map

$$\pi_1^{\text{ét}}(Y_{K_v}) \to G_v := G_{K_v} \tag{$*_v$}$$

on fundamental groups. Again every K_v-point of Y gives rise to a splitting of $(*_v)$ by functoriality, so we have a local section map $Y(K_v) \to \mathrm{Sec}(Y_{K_v}/K_v)$. Restriction to a decomposition group at v gives a map $\mathrm{Sec}(Y/K) \to \mathrm{Sec}(Y_{K_v}/K_v)$.

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Definition

An element $y \in \mathrm{Sec}(Y/K)$ (section of $(*)$) is called *locally geometric* (or *Selmer*) just when $y|_{G_v} \in \mathrm{Sec}(Y_{K_v}/K_v)$ lies in the image of the local section map for every place v of K. We write $\mathrm{Sec}(Y/K)^{\mathrm{l.g.}}$ for the set of locally geometric sections, which contains $Y(K)$.

The main theorem

If v is a finite place of K, then we have a localisation map $\mathrm{Sec}(Y/K)^{\mathrm{l.g.}} \to Y(K_v)$ sending a locally geometric section y to the unique K_v-point y_v of Y whose associated local section is $y|_{G_v}$.

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Theorem (B.–Stix, in progress)

Let K be a number field containing no CM subfield, let Y/K be a connected smooth proper curve of genus ≥ 2, and let v be a finite place of K. Then the image of the localisation map

$$\begin{aligned} \mathrm{Sec}(Y/K)^{\mathrm{l.g.}} &\to Y(K_v) \\ y &\mapsto y_v \end{aligned}$$

is finite.

Aside: The finite descent obstruction

The finite descent obstruction cuts out an intermediate set

$$Y(K) \subseteq Y(\mathbb{A}_K)_\bullet^{\mathrm{f-cov}} \subseteq Y(\mathbb{A}_K)_\bullet .$$

It is believed that in fact $Y(K) = Y(\mathbb{A}_K)_\bullet^{\mathrm{f-cov}}$.

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Theorem (Harari–Stix)

The image of the localisation map $\mathrm{Sec}(Y/K)^{\mathrm{l.g.}} \to Y(\mathbb{A}_K)_\bullet$ is the finite descent set $Y(\mathbb{A}_K)_\bullet^{\mathrm{f-cov}}$.

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Rephrasing of the main theorem

In the same setup as the main theorem, the image of the projection $Y(\mathbb{A}_K)_\bullet^{\mathrm{f-cov}} \to Y(K_v)$ is finite for all finite places v.

The Lawrence–Venkatesh method

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(without good reduction assumptions)

Associating Galois representations to points

In 2019, Lawrence and Venkatesh re-proved

Theorem (Faltings (Mordell Conjecture))

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The main idea of the proof, present already in the first proof by Faltings, is to assign a Galois representation V_y to each rational point $y \in Y(K)$, and study these instead.

Let $X \to Y$ be a smooth proper map. To a point $y \in Y(K)$, we can associate the Galois representation $\mathrm{H}^i_{\text{ét}}(X_{y,\bar{K}}, \mathbb{Q}_p)$. This representation is pure of weight i, unramified outside a fixed finite set of places of K (not depending on y).

Constraints on global representations

Lemma (Hermite–Minkowski, Faltings)

Let K be a number field and S a finite set of places of K, and $i, d \geq 0$. Then there are, up to isomorphism, only finitely many semisimple representations V of G_K of dimension d which are unramified and pure* of weight i outside S.

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*: char. poly. of geometric Frobenius has *integer* coefficients and its roots are Weil numbers of weight i.

Variation of local representations

Suppose that v is a p-adic place of K. If we have a v-adic point $y \in Y(K_v)$, then we can assign it the local Galois representation $\mathrm{H}^i_{\text{ét}}(X_{y,\bar{K}_v}, \mathbb{Q}_p)$. If y is K-rational, this is just the restriction of V_y to G_v.

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How these local representations depend on the point $y \in Y(K_v)$ is well-understood, via the theory of *period maps*.

Period maps

Theorem (Fontaine, Berger, Faltings, Tsuji, Scholze)

Let Z/K_v be smooth and proper. Then $\mathrm{H}^i_{\mathrm{dR}}(Z/K_v)$ carries both a discrete (φ, N, G_v)-module structure and a Hodge filtration. Both of these structures are determined by the Galois representation $\mathrm{H}^i_{\text{ét}}(Z_{\bar{K}_v}, \mathbb{Q}_p)$ via the comparison isomorphism $\mathrm{D}_{\mathrm{dR}}(\mathrm{H}^i_{\text{ét}}(Z_{\bar{K}_v}, \mathbb{Q}_p)) \cong \mathrm{H}^i_{\mathrm{dR}}(Z/K_v)$.

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If $y_0 \in Y(K_v)$ is a base point, then in a small neighbourhood U_{y_0} of y_0, there is a K_v-analytic *period map*

$$\Phi_{y_0} \colon U_{y_0} \to \{\text{filtrations on } \mathrm{H}^i_{\mathrm{dR}}(X_{y_0}/K_v)\},$$

such that there is an isomorphism of filtered discrete (φ, N, G_v)-modules

$$\left(\mathrm{H}^i_{\mathrm{dR}}(X_y/K_v), \mathrm{Hodge}\right) \cong \left(\mathrm{H}^i_{\mathrm{dR}}(X_{y_0}/K_v), \Phi_{y_0}(y)\right)$$

for all $y \in U_{y_0}$.

Using Galois representations to prove finiteness

Let $Y(K)_{\mathrm{ss}}$ denote the set of K-rational points such that V_y is semisimple.

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Lemma

Write $\mathcal{H}_{y_0}$ for the Zariski-closure of the image of the period map at y_0.

i) Suppose that $\dim_{\mathbb{Q}_p} \mathrm{Aut}_{(\varphi,N,G_v)}(\mathrm{H}^i_{\mathrm{dR}}(X_{y_0}/K_v)) < \dim_{K_v} \mathcal{H}_{y_0}$. Then $Y(K)_{\mathrm{ss}} \cap U_{y_0}$ is finite.

ii) Suppose that the above holds for all $y_0 \in Y(K_v)$. Then $Y(K)_{\mathrm{ss}}$ is finite.

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Proof.

For $y \in Y(K)_{\mathrm{ss}}$, Faltings' Lemma implies that there are only finitely many possibilities for the filtered discrete (φ, N, G_v)-module structure on $\mathrm{H}^i_{\mathrm{dR}}(X_y/K_v)$. This says that $\Phi_{y_0}(y)$ lies in a finite number of $\mathrm{Aut}_{(\varphi,N,G_v)}(\mathrm{H}^i_{\mathrm{dR}}(X_{y_0}/K_v))$-orbits – a proper subspace of $\mathcal{H}_{y_0}$.

This implies that there is a non-zero rational function α on $\mathcal{H}_{y_0}$ which vanishes on all of these orbits. Thus $\alpha \circ \Phi_{y_0}$ is a non-zero meromorphic function on the disc U_{y_0} vanishing on $Y(K)_{\mathrm{ss}} \cap U_{y_0}$. □

Establishing the dimension inequality

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For an upper bound on the dimension of the automorphism group:

Lemma

Suppose that X_{y_0} is defined over a finite extension L/K. Then

$$\dim_{\mathbb{Q}_p}\mathrm{Aut}_{(\varphi,N,G_v)}(\mathrm{H}^i_{\mathrm{dR}}(X_{y_0}/K_v)) \leq n\cdot\left(\dim_L \mathrm{H}^i_{\mathrm{dR}}(X_{y_0}/L)\right)^2 ,$$

where n is the number of places of L over v.

Example: abelian schemes over finite extensions

Suppose now that L/K is a finite extension, $X \to Y_L$ is a polarised abelian scheme of dimension g, and $i = 1$. For "generic" X, we would expect $\dim_{K_v} \mathcal{H}_{y_0} = [L : K] \cdot \frac{g(g+1)}{2}$. On the other hand, the lemma gives

$$\dim_{\mathbb{Q}_p} \mathrm{Aut}_{(\varphi,N,G_v)}(\mathrm{H}^1_{\mathrm{dR}}(X_{y_0}/K_v)) \leq n \cdot 4g^2 \,.$$

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Consequence

Suppose that X is "generic", that $[L : K] \geq 8$, and that v does not split in L. Then $Y(K)_{\mathrm{ss}}$ is finite.

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Remark: Faltings' (hard) proof of the Tate Conjecture for abelian varieties shows that $Y(K)_{\mathrm{ss}} = Y(K)$.

Refinement: abelian-by-finite families

Definition

An *abelian-by-finite family* over Y is a sequence $X \to Y' \to Y$ with $Y' \to Y$ a finite étale covering and $X \to Y'$ a polarised abelian scheme.

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The fibres of an abelian-by-finite family are disjoint unions of polarised abelian varieties. The cohomology of the fibres thus has a decomposition

$$\mathrm{H}^1_{\text{ét}}(X_{y,\bar{K}}, \mathbb{Q}_p) \cong \bigoplus_{y' \in |Y'_y|} \mathrm{Ind}_{G_{K(y')}}^{G_K} \mathrm{H}^1_{\text{ét}}(X_{y',\bar{K}(y')}, \mathbb{Q}_p) \, .$$

The Kodaira–Parshin family

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The Kodaira–Parshin family

In the argument of Lawrence–Venkatesh, the abelian-by-finite family $X \to Y' \xrightarrow{\pi} Y$ and finite place v are chosen to satisfy three technical conditions:

A) The restriction of v to any CM subfield of K is invariant under the conjugation.

B) The family $X \to Y' \to Y$ has "full monodromy".

C) For every point $y \in Y(K_v)$, the number of elements of $Y'_y(\bar{K}_v)$ contained in a G_v-orbit of size ≥ 8 is $> \frac{d}{d+1} \cdot \deg(\pi)$, where $d > 0$ is the relative dimension of $X \to Y'$.

The Principal Dichotomy

Theorem (Principal Dichotomy)

Let v be a p-adic place of K and $X \to Y' \to Y$ an abelian-by-finite family over Y. Suppose that conditions (A) and (C) are satisfied. Then for every point $y \in Y(K)$ there is a closed point y' of Y'_y and a place w of $L = K(y')$ over v such that $[L_w : K_v] \geq 8$ and either:

a) the representation $\mathrm{H}^1_{\text{ét}}(X_{y',\bar{L}}, \mathbb{Q}_p)$ is simple; or
b) $\mathrm{H}^1_{\mathrm{dR}}(X_{y',\bar{L}_w}, \mathbb{Q}_p)$ has a non-zero filtered (φ, N, G_w)-submodule W such that $\dim_{L_w} \mathrm{F}^1 W \geq \frac{1}{2} \dim_{L_w} W$.

The proof of Mordell (outline)

We need to prove finiteness of the sets $Y(K)_{(a)}$ and $Y(K)_{(b)}$ of K-rational points satisfying (a) and (b), respectively.

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For points of type (a), a dimension-count (similar to the one on an earlier slide) shows that the image of $Y(K)_{(a)} \cap U_{y_0}$ under the period map at $y_0 \in Y(K_v)$ is not Zariski-dense in $\mathcal{H}_{y_0}$, and we obtain finiteness of $Y(K)_{(a)}$ as before.

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For points of type (b), condition (b) directly implies that the image of $Y(K)_{(b)} \cap U_{y_0}$ is contained in a proper Zariski-closed subspace of $\mathcal{H}_{y_0}$, and we obtain finiteness of $Y(K)_{(b)}$ as before.

Lawrence–Venkatesh for Galois sections

Recap of the main theorem

Theorem

Let K be a number field, let Y/K be a connected smooth proper curve of genus ≥ 2, and let v be a finite place of K satisfying condition (A). Then the image of the localisation map

$$\begin{aligned} \mathrm{Sec}(Y/K)^{\mathrm{l.g.}} &\to Y(K_v) \\ y &\mapsto y_v \end{aligned}$$

is finite.

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is finite.

The proof follows the same outline as Lawrence–Venkatesh.

Adapting the Lawrence–Venkatesh method

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i) Given a smooth proper family $X \to Y$, we need to describe a way to assign Galois representations V_y to elements $y \in \mathrm{Sec}(Y/K)^{\mathrm{l.g.}}$, such that:
   - V_y is unramified and pure of weight i outside a fixed finite set of places of K;
   - V_y is de Rham at places over p; and
   - the restriction $V_y|_{G_v}$ is controlled by the period map associated to $X \to Y$.

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ii) We need to find an abelian-by-finite family $X \to Y' \to Y$ satisfying conditions (B) and (C) (for our given v).

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ii) We need to find an abelian-by-finite family $X \to Y' \to Y$ satisfying conditions (B) and (C) (for our given v).

iii) We need a version of the Principal Dichotomy for the Galois representations arising from elements of $\mathrm{Sec}(Y/K)^{\mathrm{l.g.}}$.

Associating Galois representations to sections

Let $f\colon X \to Y$ be a smooth proper map. The relative étale cohomology $\mathrm{R}^i_{\text{ét}} f_* \underline{\mathbb{Q}}_p$ is a $\mathbb{Q}_p$-local system on Y, and hence corresponds to a representation V of the étale fundamental group $\pi_1^{\text{ét}}(Y)$. Given a section y of the structure map $\pi_1^{\text{ét}}(Y) \to G_K$, we may restrict the action on V to make it a representation of G_K, which we denote by V_y.

If the section y is locally geometric, so $y|_{G_u}$ is the section arising from a point $y_u \in Y(K_u)$ for all places u of K, then $V_y|_{G_u} \cong \mathrm{H}^i_{\text{ét}}(X_{y_u,\bar{K}_u}, \mathbb{Q}_p)$ for all places u.

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- V_y is unramified outside a fixed set of places of K (depending only on $X \to Y$), and is pure of weight i outside that set.
- V_y is de Rham at all p-adic places of K, and its restriction to G_v is controlled by the v-adic period map; and

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- V_y is de Rham at all p-adic places of K, and its restriction to G_v is controlled by the v-adic period map; and

Thus the representations V_y are well-behaved enough to run the Lawrence–Venkatesh argument more-or-less verbatim.

General philosophy

- Several approaches to Diophantine geometry (Lawrence–Venkatesh, Chabauty–Kim,. . .) revolve around assigning Galois representations to rational points on a variety Y, by taking fibres of a local system on Y.

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- In this setup, we can assign Galois representations not just to rational points, but also to Galois sections.
- These methods then constrain not just the set $Y(K) \subseteq Y(K_v)$, but even the image of $\mathrm{Sec}(Y/K)^{\mathrm{l.g.}} \to Y(K_v)$.

Example (speculative)

Let $Y/\mathbb{Q}$ be a connected smooth proper curve of genus ≥ 2, with a rational point b. If p is a prime of good reduction, then the Chabauty–Kim method gives a nested sequence of subsets

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Consequence: We can give examples of curves $Y/\mathbb{Q}$ and primes p for which the image of the map $\mathrm{Sec}(Y/\mathbb{Q})^{\mathrm{l.g.}} \to Y(\mathbb{Q}_p)$ is *exactly* $Y(\mathbb{Q})$.