KÄHLER GROUPS AND THE NON-ABELIAN HODGE CORRESPONDENCE

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[These are the notes for the talk I gave in the Harvard study seminar on the non-abelian Hodge correspondence. Most of this is drawn from §4 of Simpson's paper [1].]

1. Introduction

A very general question arising in topology is which groups can arise as fundamental groups of certain classes of topological spaces. Unless the class of topological spaces under consideration is quite restrictive, this kind of question tends to have a rather uninteresting answer. For instance, any finitely presented group is the fundamental group of a compact manifold[1], and is even the fundamental group of a compact fourfold. So to get an interesting answer to this question, one wants to look at more restrictive classes of topological spaces.

Definition 1.1. A *Kähler manifold* (X, ω) is a complex manifold X endowed with a Hermitian metric whose associated real $(1,1)$-form ω is closed.

Example 1.2.
- Projective space $X = \mathbb{P}^N(\mathbb{C})$ with the Fubini–Study metric ω is Kähler.
- If X is a complex projective variety embedded in $\mathbb{P}^N_{\mathbb{C}}$, then $X(\mathbb{C})$ is a Kähler manifold with respect to the restriction of the Fubini–Study metric.
- A compact Kähler manifold (X, ω) is a projective variety if and only if ω is integral, i.e. its cohomology class $[\omega] \in \mathrm{H}^2(X, \mathbb{R})$ lies in $\mathrm{H}^2(X, \mathbb{Z})$.

Definition 1.3. A finitely presented group Γ is *Kähler* just when it is isomorphic to the fundamental group of a compact Kähler manifold.

Example 1.4. The group $\mathbb{Z}^n$ is Kähler if and only if n is even.

Proof. In one direction, if n is even then $\mathbb{Z}^n$ is the fundamental group of an abelian variety of dimension $n/2$.

Conversely, if $\mathbb{Z}^n = \pi_1(X)$ is Kähler, then by the Hurewicz Theorem we know that $\mathrm{H}_1(X, \mathbb{Z}) = \mathbb{Z}^n$, and hence $\mathrm{H}^1(X, \mathbb{C}) = \mathbb{C}^n$. But we have the Hodge decomposition

$$\mathrm{H}^1(X, \mathbb{C}) = \mathrm{H}^{1,0}(X, \mathbb{C}) \oplus \mathrm{H}^{0,1}(X, \mathbb{C})$$

and the two factors on the right-hand side are interchanged under complex conjugation, so have the same dimension. Thus

$$n = \dim_{\mathbb{C}} \mathrm{H}^1(X, \mathbb{C}) = 2 \dim_{\mathbb{C}} \mathrm{H}^{1,0}(X, \mathbb{C})$$

is even. □

[1]For this talk, manifold always means manifold without boundary.

This gives a first indication that Hodge theory can help constrain the class of Kähler groups, giving us criteria to show that certain groups are *not* Kähler. Today, our aim is to explain how one can use the non-abelian Hodge correspondence to prove the following.

Theorem 1.5. $\mathrm{SL}_n(\mathbb{Z})$ *is not Kähler for* $n \geq 3$.

2. Variations of Hodge structure

Let's now recap a few definitions from the previous talks, beginning with the slightly non-standard notion of a complex Hodge structure.

Definition 2.1. A $\mathbb{C}$-Hodge structure (pure of weight k), or $\mathbb{C}$-HS for short, is a finite-dimensional $\mathbb{C}$-vector space V endowed with a $\mathbb{C}$-linear decomposition

$$V = \bigoplus_{p+q=k} V^{p,q} .$$

A *polarisation* on V is a pairing

$$Q\colon V \times V \to \mathbb{C} ,$$

which is required to satisfy:

- (sesquilinearity) Q is bilinear, and satisfies $Q(\lambda\alpha, \mu\beta) = \lambda\bar{\mu}Q(\alpha, \beta)$ for all $\lambda, \mu \in \mathbb{C}$ and $\alpha, \beta \in V$;
- ($(-1)^k$-conjugate symmetry) $Q(\beta, \alpha) = (-1)^k\overline{Q(\alpha, \beta)}$ for all $\alpha, \beta \in V$;
- the decomposition $V = \bigoplus_{p+q=k} V^{p,q}$ is orthogonal with respect to Q, i.e. $Q(\alpha, \beta) = 0$ if $\alpha \in V^{p,q}$ and $\beta \in V^{p',q'}$ with $(p, q) \neq (p', q')$;
- $i^{p-q}Q$ is positive-definite on $V^{p,q}$, i.e. $i^{p-q}Q(\alpha, \alpha) > 0$ for all non-zero $\alpha \in V^{p,q}$. (In particular, this implies that $i^{p-q}Q(\alpha, \alpha)$ is real.)

We sometimes consider Q as a $\mathbb{C}$-linear map $V \otimes_{\mathbb{C}} \bar{V} \to \mathbb{C}$ where $\bar{V}$ denotes V with the conjugate complex structure.

Given a polarisation Q on V, one can associate a positive-definite Hermitian form K, by taking the orthogonal direct sum of the pairings $i^{p-q}Q$ on each $V^{p,q}$. So, if s denotes the $\mathbb{C}$-linear involution of V which acts by multiplication by $(-1)^p$ on each $V^{p,q}$, then the Hermitian metric K and the polarisation Q are related by

$$K(\alpha, \beta) = i^{-k}Q(\alpha, s(\beta)) .$$

Example 2.2. If (X, ω) is a compact Kähler manifold of dimension n, then the Hodge decomposition

$$\mathrm{H}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \mathrm{H}^{p,q}(X, \mathbb{C})$$

makes $\mathrm{H}^k(X, \mathbb{C})$ into a $\mathbb{C}$-HS (of weight k) for all k.

These Hodge structures can be decomposed further. Recall that the *Lefschetz operator* $L\colon \mathrm{H}^k(X, \mathbb{C}) \to \mathrm{H}^{k+2}(X, \mathbb{C})$ is the map given by cupping with the Kähler class $[\omega]$. If $k \leq n$, the map

$$L^{n-k}\colon \mathrm{H}^k(X, \mathbb{C}) \xrightarrow{\sim} \mathrm{H}^{2n-k}(X, \mathbb{C})$$

is a $\mathbb{C}$-linear isomorphism (hard Lefschetz), and we define the *primitive part* $\mathrm{H}^k(X, \mathbb{C})_{\mathrm{prim}}$ is, by definition, the kernel of L^{n-k+1} on $\mathrm{H}^k(X, \mathbb{C})$. The primitive cohomology is a

Hodge substructure of $\mathrm{H}^k(X,\mathbb{C})$, and the decomposition

$$\mathrm{H}^k(X,\mathbb{C}) = \bigoplus_{j\geq 0} L^j\mathrm{H}^{k-2j}(X,\mathbb{C})_{\mathrm{prim}}$$

coming from hard Lefschetz is a decomposition into Hodge substructures.

Both of the above examples are actually *polarised* Hodge structures. In the case of the primitive cohomology, one takes the polarisation given by

$$Q(\alpha,\beta) = \int_X \omega^{n-k}\wedge\alpha\wedge\beta\,.$$

As for the polarisation on $\mathrm{H}^k(X,\mathbb{C})$, we take the orthogonal direct sum of the pairings given by[2]

$$Q(\alpha,\beta) = (-1)^j\int_X \omega^{n-k}\wedge\alpha\wedge\beta$$

on $L^j\mathrm{H}^{k-2j}(X,\mathbb{C})_{\mathrm{prim}}$.

Remark 2.3. One can also study Hodge structures with coefficients in rings other than $\mathbb{C}$, usually either $\mathbb{R}$, $\mathbb{Q}$ or $\mathbb{Z}$. For $R\in\{\mathbb{R},\mathbb{Q},\mathbb{Z}\}$, an *$R$-Hodge structure* (pure of weight k) is a finitely generated R-module V together with a decomposition

$$\mathbb{C}\otimes_R V = \bigoplus_{p+q=k} V^{p,q}$$

satisfying the additional condition that $\overline{V^{p,q}} = V^{q,p}$. A *polarisation* of an R-HS is an R-bilinear pairing $Q\colon V\otimes_R V\to R$ such that the induced conjugate-linear pairing on $\mathbb{C}\otimes_R V$ is a polarisation in the above sense.

All of the above examples are actually $\mathbb{R}$-Hodge structures (or more precisely, they make $\mathrm{H}^k(X,\mathbb{R})$ and $\mathrm{H}^k(X,\mathbb{R})_{\mathrm{prim}}$ into $\mathbb{R}$-Hodge structures), and the polarisations are defined over $\mathbb{R}$. If X is a complex projective variety (i.e. ω is integral), then these are all defined over $\mathbb{Z}$.

2.1. **Variations.** Now let X be a complex manifold. Write $\mathcal{A}$ for the sheaf of C^∞ complex functions, $\mathcal{A}^k$ for the sheaf of C^∞ complex k-forms, and $\mathcal{A}^{p,q}$ for the sheaf of complex (p,q)-forms on X. We will follow the convention that complex local systems on X are denoted by blackboard bold characters like $\mathbb{E}$, and complex C^∞ flat bundles are denoted by calligraphic characters like $\mathcal{E}$ (or $(\mathcal{E},D)$ if we want to name the connection explicitly). In case we forget to say it, $\mathcal{E}$ will always be the flat bundle corresponding to $\mathbb{E}$ under the Riemann–Hilbert correspondence, and so on.

We can make sense of what is meant by a "family of Hodge structures" over the base X.

Definition 2.4. A *variation of $\mathbb{C}$-Hodge structure* (pure of weight k), or $\mathbb{C}$-VHS for short, is a $\mathbb{C}$-local system $\mathbb{E}$ on X endowed with an $\mathcal{A}$-linear decomposition

$$\mathcal{E} = \bigoplus_{p+q=k}\mathcal{E}^{p,q}$$

of the associated flat bundle $\mathcal{E}:=\mathcal{A}\otimes_{\mathbb{C}}\mathbb{E}$. This decomposition is required to satisfy the *Griffiths transversality condition*

$$D(\mathcal{E}^{p,q})\subseteq(\mathcal{E}^{p,q}\otimes_{\mathcal{A}}\mathcal{A}^{1,0})\oplus(\mathcal{E}^{p,q}\otimes_{\mathcal{A}}\mathcal{A}^{0,1})\oplus(\mathcal{E}^{p-1,q+1}\otimes_{\mathcal{A}}\mathcal{A}^{1,0})\oplus(\mathcal{E}^{p+1,q-1}\otimes_{\mathcal{A}}\mathcal{A}^{0,1})\,.$$

[2] I'm a little confused about the sign here: I seem to get that the positive-definiteness condition requires no sign $(-1)^j$.

Let's explain this condition in words. The connection

$$D\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{A}^1$$

can be decomposed into components according to the Hodge decompositions of $\mathcal{E}$ and $\mathcal{A}^1$. The Griffiths transversality condition is saying that most of these components are zero: the only components that can appear are the ones going from (p,q) to $(p,q,1,0)$, $(p,q,0,1)$, $(p-1,q+1,1,0)$ and $(p+1,q-1,0,1)$. Note that it we required more strongly that

$$D(\mathcal{E}^{p,q}) \subseteq (\mathcal{E}^{p,q} \otimes_{\mathcal{A}} \mathcal{A}^{1,0}) \oplus (\mathcal{E}^{p,q} \otimes_{\mathcal{A}} \mathcal{A}^{0,1})\,,$$

then we would be saying that the decomposition of $\mathcal{E}$ is compatible with the connection, i.e. $\mathcal{E}$ is a direct sum of flat bundles $\mathcal{E}^{p,q}$. Roughly speaking, Griffiths transversality is saying that the decomposition of $\mathcal{E}$ is "close" to being compatible with connections.

A *polarisation* on $\mathbb{E}$ is a $\mathbb{C}$-linear pairing

$$Q\colon \mathbb{E} \otimes_{\mathbb{C}} \bar{\mathbb{E}} \to \underline{\mathbb{C}}$$

of $\mathbb{C}$-local systems for which the corresponding pairing

$$Q\colon \mathcal{E} \otimes_{\mathcal{A}} \bar{\mathcal{E}} \to \mathcal{A}$$

of flat bundles satisfies

- $Q(\beta,\alpha) = (-1)^k\overline{Q(\alpha,\beta)}$ for all sections α,β of $\mathcal{E}$;
- the decomposition $\mathcal{E} = \bigoplus_{p+q=k} \mathcal{E}^{p,q}$ is orthogonal with respect to Q;
- $i^{p-q}Q$ is real-valued and positive-definite on $\mathcal{E}^{p,q}$, i.e. $i^{p-q}Q(\alpha,\alpha) > 0$ is a positive real-valued C^∞ function for all non-vanishing sections α of $\mathcal{E}^{p,q}$.

In other words, the restriction of Q to each fibre of $\mathbb{E}$ should be a polarisation of the Hodge structure $\mathbb{E}_x$.

As before, the pairing

$$K\colon \mathcal{E} \otimes_{\mathcal{A}} \bar{\mathcal{E}} \to \mathcal{A}$$

given by i^{p-q} on $\mathcal{E}^{p,q}$ is a positive-definite Hermitian form, i.e. is a C^∞ Hermitian metric on the flat bundle $\mathcal{E}$.

Remark 2.5. If $\mathbb{E}$ is a (polarised) $\mathbb{C}$-VHS, then the decomposition of $\mathcal{E}$ induces a (polarised) $\mathbb{C}$-HS on each fibre $\mathbb{E}_x = \mathcal{E}_x$.

Lemma 2.6. *Let $\mathbb{E}$ be a $\mathbb{C}$-PVHS and let $\mathbb{E}_0 \leq \mathbb{E}$ be a sub-$\mathbb{C}$-VHS, with annihilator $\mathbb{E}_0^\perp$ with respect to the polarisation Q (this is also the orthogonal complement with respect to the Hermitian metric K). Then $\mathbb{E}_0^\perp$ is also a sub-$\mathbb{C}$-VHS, and is a complement to $\mathbb{E}_0$, meaning that we have a direct sum decomposition*

$$\mathbb{E} = \mathbb{E}_0 \oplus \mathbb{E}_0^\perp\,.$$

When we equip $\mathbb{E}_0$ and $\mathbb{E}_0^\perp$ with the restriction of the polarisation Q, the above becomes an orthogonal decomposition of $\mathbb{C}$-PVHS.

Proof. The fact that $\mathbb{E}_0$ is a sub-$\mathbb{C}$-VHS means that its associated flat bundle $\mathcal{E}_0$ decomposes as

$$\mathcal{E}_0 = \bigoplus_{p+q=k} \mathcal{E}_0^{p,q}\,,$$

where each $\mathcal{E}_0^{p,q}$ is an $\mathcal{A}$-submodule of $\mathcal{E}^{p,q}$ (note that the Griffiths transversality condition on $\mathcal{E}_0$ is automatic from). Since the polarisation is, up to a scalar, positive-definite on each $\mathcal{E}^{p,q}$, this means that the annihilator of $\mathcal{E}_0$ also decomposes as

$$\mathcal{E}_0^\perp = \bigoplus_{p+q=k} (\mathcal{E}_0^\perp)^{p,q} ,$$

where

$$\mathcal{E}_0^{p,q} \oplus (\mathcal{E}_0^\perp)^{p,q} = \mathcal{E}^{p,q} .$$

Thus $\mathcal{E}_0^\perp$ is a sub-$\mathbb{C}$-VHS and $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_0^\perp$, so $\mathbb{E} = \mathbb{E}_0 \oplus \mathbb{E}_0^\perp$. □

Now if $\mathbb{E}$ is a $\mathbb{C}$-VHS, then by Griffiths transversality we can uniquely decompose the connection D on the associated flat bundle $\mathcal{E}$ as

$$D = \partial + \bar\partial + \theta + \bar\theta \tag{1}$$

where ∂, $\bar\partial$, θ and $\bar\theta$ restrict to the components $\mathcal{E}^{p,q}$ as

$$\begin{aligned}
\partial &\colon \mathcal{E}^{p,q} \to \mathcal{E}^{p,q} \otimes_{\mathcal{A}} \mathcal{A}^{1,0} \\
\bar\partial &\colon \mathcal{E}^{p,q} \to \mathcal{E}^{p,q} \otimes_{\mathcal{A}} \mathcal{A}^{0,1} \\
\theta &\colon \mathcal{E}^{p,q} \to \mathcal{E}^{p-1,q+1} \otimes_{\mathcal{A}} \mathcal{A}^{1,0} \\
\bar\theta &\colon \mathcal{E}^{p,q} \to \mathcal{E}^{p+1,q-1} \otimes_{\mathcal{A}} \mathcal{A}^{0,1} .
\end{aligned}$$

It follows from the Leibniz rule for D that ∂ and $\bar\partial$ are connections of type $(1,0)$ and $(0,1)$, i.e. satisfy

$$\partial(f\alpha) = \partial(f)\alpha + f\partial(\alpha) \quad \text{and} \quad \bar\partial(f\alpha) = \bar\partial(f)\alpha + f\bar\partial(\alpha) ,$$

while the maps θ and $\bar\theta$ are $\mathcal{A}$-linear.

Lemma 2.7. *Suppose that $\mathbb{E}$ is a $\mathbb{C}$-PVHS. Then the Hermitian form K satisfies*

$$\begin{aligned}
K(\partial\alpha,\beta) + K(\alpha,\bar\partial\beta) &= \partial K(\alpha,\beta) & \qquad K(\theta\alpha,\beta) &= K(\alpha,\bar\theta\beta) \\
K(\bar\partial\alpha,\beta) + K(\alpha,\partial\beta) &= \bar\partial K(\alpha,\beta) & \qquad K(\bar\theta\alpha,\beta) &= K(\alpha,\theta\beta)
\end{aligned}$$

for all sections α,β of $\mathcal{E}$. (Here, we extend K to $\mathcal{E}$-valued forms in the obvious way. For instance, if η is a section of $\mathcal{A}^1$, then $K(\alpha,\beta\otimes\eta) \coloneqq K(\alpha,\beta)\bar\eta$.)

Proof. Since the polarisation $Q\colon \mathcal{E} \otimes_{\mathcal{A}} \bar{\mathcal{E}} \to \mathcal{A}$ comes from a morphism of local systems, it is flat and so we have the identity

$$Q(D\alpha,\beta) + Q(\alpha,D\beta) = \mathrm{d}Q(\alpha,\beta) .$$

Substituting $D = \partial + \bar\partial + \theta + \bar\theta$ and breaking up by type, this shows that

$$\begin{aligned}
Q(\partial\alpha,\beta) + Q(\alpha,\bar\partial\beta) &= \partial Q(\alpha,\beta) & \qquad Q(\theta\alpha,\beta) + Q(\alpha,\bar\theta\beta) &= 0 \\
Q(\bar\partial\alpha,\beta) + Q(\alpha,\partial\beta) &= \bar\partial Q(\alpha,\beta) & \qquad Q(\bar\theta\alpha,\beta) + Q(\alpha,\theta\beta) &= 0 .
\end{aligned}$$

(These are easiest to check first for α and β sections of $\mathcal{E}^{p,q}$ and $\mathcal{E}^{p',q'}$, respectively.) These identities imply the claimed identities for the Hermitian metric K. □

Corollary 2.8. *Let $\mathbb{E}$ be a $\mathbb{C}$-PVHS on a compact Kähler manifold X. Then the Hermitian metric K associated to the polarisation Q is harmonic.*

Proof. Recall that a flat connection D on a C^∞ vector bundle $\mathcal{E}$ can be decomposed uniquely as

$$D = \partial_K + \bar\partial + \theta + \bar\theta_K$$

for maps

$$\begin{aligned} \partial_K &\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{A}^{1,0} \\ \bar\partial &\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{A}^{0,1} \\ \theta &\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{A}^{1,0} \\ \bar\theta_K &\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{A}^{0,1}, \end{aligned}$$

where ∂_K and $\bar\partial$ are connections of types $(1,0)$ and $(0,1)$, θ and $\bar\theta_K$ are $\mathcal{A}$-linear, and we have the compatibility conditions

$$K(\bar\partial\alpha,\beta) + K(\alpha,\partial_K\beta) = \bar\partial K(\alpha,\beta) \quad \text{and} \quad K(\theta\alpha,\beta) = K(\alpha,\bar\theta_K\beta)$$

with respect to the metric K. The metric is harmonic if and only if $(\bar\partial+\theta)^2 = 0$.

In our case, Lemma 2.7 shows that $\partial_K = \partial$ and $\bar\theta_K = \bar\theta$. So to check K is harmonic, we want to verify that $(\bar\partial+\theta)^2 = 0$. Since $D = \partial + \bar\partial + \theta + \bar\theta$, it follows that

$$\begin{aligned} \bar\partial^2 &\colon \mathcal{E}^{p,q} \to \mathcal{E}^{p,q} \otimes_{\mathcal{A}} \mathcal{A}^{0,2} \\ \bar\partial\theta + \theta\bar\partial &\colon \mathcal{E}^{p,q} \to \mathcal{E}^{p-1,q+1} \otimes_{\mathcal{A}} \mathcal{A}^{1,1} \\ \theta^2 &\colon \mathcal{E}^{p-2,q+2} \to \mathcal{E}^{p-2,q+2} \otimes_{\mathcal{A}} \mathcal{A}^{2,0} \end{aligned}$$

are the corresponding components of $D^2 \colon \mathcal{E} \to \mathcal{E}\otimes_{\mathcal{A}}\mathcal{A}^2$, so are all zero since $D^2 = 0$. This proves that $\bar\partial^2 = 0$, $\bar\partial\theta + \theta\bar\partial = 0$ and $\theta^2 = 0$, so K is harmonic. □

Corollary 2.9. *Let $\mathbb{E}$ be a $\mathbb{C}$-PVHS on a compact Kähler manifold X. Then $\mathbb{E}$ is a semisimple local system.*

Proof. A local system is semisimple if and only if its associated flat bundle $\mathcal{E}$ admits a harmonic metric [1, Theorem 1]. □

In what comes, we will need to know another fact about polarised variations of Hodge structure, which despite its simple statement is remarkably hard to prove.

Theorem 2.10 (Theorem of the fixed part[3])**.** *Suppose that X is compact, and let $\mathbb{E}$ be a $\mathbb{C}$-PVHS. Then the global sections*

$$\mathrm{H}^0(X,\mathbb{E}) = \mathrm{H}^0(X,\mathcal{E})^{D=0}$$

of $\mathbb{E}$ is a $\mathbb{C}$-HS, where the Hodge decomposition is the restriction of the decomposition

$$\mathrm{H}^0(X,\mathcal{E}) = \bigoplus_{p+q=k} \mathrm{H}^0(X,\mathcal{E}^{p,q}).$$

Proof. We know that $\mathrm{H}^0(X,\mathbb{E})$ is finite-dimensional, so the claim at issue is that it is a *graded* subspace of $\mathrm{H}^0(X,\mathcal{E})$. In other words, let $\alpha \in \mathrm{H}^0(X,\mathcal{E})^{D=0}$ be a flat global section of $\mathcal{E}$, and write

$$\alpha = \sum_{p+q=k} \alpha^{p,q}$$

for global sections $\alpha^{p,q}$ of $\mathcal{E}^{p,q}$. We want to show that each $\alpha^{p,q}$ is flat.

[3] Cite "curvature properties of the hodge bundles"

We proceed by induction. Suppose that $\alpha^{p-1,q+1}$ is flat. Taking the $(p+1,q-1,0,1)$- and $(p,q,1,0)$-components of $D(\alpha)=0$ shows that

$$\bar{\partial}(\alpha^{p,q})=0 \quad \text{and} \quad \theta(\alpha^{p,q})=0\,.$$

Using this and Lemma 2.7, we compute

$$\begin{aligned}\partial\bar{\partial}K(\alpha^{p,q},\alpha^{p,q}) &= \partial K(\alpha^{p,q},\partial\alpha^{p,q})\\ &= K(\partial\alpha^{p,q},\partial\alpha^{p,q})+K(\alpha^{p,q},\bar{\partial}\partial\alpha^{p,q})\,.\end{aligned}$$

Now by considering the component of $D^2=0$ mapping $\mathcal{E}^{p,q}$ to $\mathcal{E}^{p,q}\otimes_{\mathcal{A}}\mathcal{A}^{1,1}$, we have the identity

$$\partial\bar{\partial}+\bar{\partial}\partial+\theta\bar{\theta}+\bar{\theta}\theta=0\,,$$

and so $\bar{\partial}\alpha^{p,q}+\theta\bar{\theta}\alpha^{p,q}=0$. So we have

$$i\partial\bar{\partial}K(\alpha^{p,q},\alpha^{p,q})=iK(\partial\alpha^{p,q},\partial\alpha^{p,q})-iK(\bar{\theta}\alpha^{p,q},\bar{\theta}\alpha^{p,q})\,.$$

On the right-hand side, both of the C^∞ forms $iK(\partial\alpha^{p,q},\partial\alpha^{p,q})$ and $-iK(\bar{\theta}\alpha^{p,q},\bar{\theta}\alpha^{p,q})$ are semi-positive (meaning that they are a non-negative $\mathbb{R}$-linear combination of differentials locally of the form $i\mathrm{d}z\wedge\mathrm{d}\bar{z}$, multiplied by non-negative-valued C^∞ functions). So $i\partial\bar{\partial}K(\alpha^{p,q},\alpha^{p,q})$ is also semi-positive. This means (by definition) that the C^∞ function $K(\alpha^{p,q},\alpha^{p,q})$ is *plurisubharmonic*. But the only plurisubharmonic functions on a compact manifold are constants, so we obtain that

$$iK(\partial\alpha^{p,q},\partial\alpha^{p,q})=-iK(\bar{\theta}\alpha^{p,q},\bar{\theta}\alpha^{p,q})=0\,.$$

Since K is positive-definite, this means that we must have $\partial\alpha^{p,q}=0$ and $\bar{\theta}\alpha^{p,q}=0$, so $D(\alpha^{p,q})=0$ and $\alpha^{p,q}$ is flat. This completes the induction. □

3. Groups of Hodge type

In this section, we will study more carefully the monodromy representation

$$\rho\colon \pi_1(X)\to \mathrm{GL}(\mathbb{E}_x)(\mathbb{C})$$

associated to a $\mathbb{C}$-PVHS $\mathbb{E}$ on a compact Kähler manifold X. Let $G\leq \mathrm{GL}(\mathbb{E}_x)$ be the complex monodromy group, i.e. the Zariski-closure of $\mathrm{im}(\rho)$.

Lemma 3.1. *G is reductive.*

Proof. We saw in Corollary 2.9 that ρ is a semisimple representation. The monodromy group of any semisimple representation is always reductive. □

There are two other interesting groups attached to the representation ρ. On the one hand, we define the *real monodromy group* $W\leq \mathrm{Res}^{\mathbb{C}}_{\mathbb{R}}\mathrm{GL}(\mathbb{E}_x)$ to be the Zariski-closure of $\mathrm{im}(\rho)$ inside the Weil restriction $\mathrm{Res}^{\mathbb{C}}_{\mathbb{R}}\mathrm{GL}(\mathbb{E}_x)$. On the other, we define the *unitary subgroup* $U\leq \mathrm{Res}^{\mathbb{C}}_{\mathbb{R}}\mathrm{GL}(\mathbb{E}_x)$ to be the intersection of $\mathrm{Res}^{\mathbb{C}}_{\mathbb{R}}G$ with the group $U(\mathbb{E}_x,K_x)\leq \mathrm{Res}^{\mathbb{C}}_{\mathbb{R}}\mathrm{GL}(\mathbb{E}_x)$ of matrices preserving the Hermitian form K_x. Since $U(\mathbb{E}_x,K_x)$ is compact[4], it follows that U is compact.

Lemma 3.2. *U is a compact real form of G.*

[4]Meaning that its $\mathbb{R}$-points are a compact group.

Before we get to the proof, let's recall what is meant by a real form. If G is an affine algebraic group over $\mathbb{C}$, then a *real form* of G is a real algebraic subgroup $G_0 \leq \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}} G$ such that the induced map $G_{0,\mathbb{C}} \to G$ is an isomorphism. If G_0 is a real form of G, then there is an associated automorphism σ of $\mathrm{Res}_{\mathbb{R}}^{\mathbb{C}} G = \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}} G_{0,\mathbb{C}}$. On A-points for an $\mathbb{R}$-algebra A, this is just the automorphism of

$$(\mathrm{Res}_{\mathbb{R}}^{\mathbb{C}} G)(A) = G_0(\mathbb{C} \otimes_{\mathbb{R}} A)$$

given by the conjugation on $\mathbb{C} \otimes_{\mathbb{R}} A$. We call σ the *conjugation* associated to the real form G_0. It is an involution of $\mathrm{Res}_{\mathbb{R}}^{\mathbb{C}} G$, and acts $\mathbb{C}$-antilinearly on the tangent bundle of $\mathrm{Res}_{\mathbb{R}}^{\mathbb{C}} G$ (which has a canonical complex structure). The real form G_0 is then the fixed locus of σ. This gives us a characterisation of the real forms of G, which is essentially just Galois descent.

Lemma 3.3. *For an affine algebraic group G over $\mathbb{C}$, the set of real forms of G is in bijection with the set of conjugations on G (involutions which act antilinearly on the tangent bundle).*

A *compact real form* of G is a real form U such that:

- $U(\mathbb{R})$ is compact; and
- $U(\mathbb{R})$ meets every connected component of $G(\mathbb{C})$ (equivalently, the scheme of connected components of U is a disjoint union of copies of $\mathrm{Spec}(\mathbb{R})$.)

A theorem of Cartan implies that any complex reductive group G has a compact real form, unique up to conjugation.

Example 3.4. Fix a Hermitian metric K on $\mathbb{C}^n$, and let $U(\mathbb{C}^n, K) \leq \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}} \mathrm{GL}_{n,\mathbb{C}}$ be the unitary group associated to K. Then $U(\mathbb{C}^n, K)$ is a real form of $\mathrm{GL}_{n,\mathbb{C}}$; the corresponding conjugation τ is given by

$$\tau(g) := (g^\dagger)^{-1} ,$$

where $g^\dagger$ denotes the Hermitian adjoint with respect to K. This is a compact real form.

Now let us prove Lemma 3.2. Let τ denote the conjugation on $\mathrm{GL}(\mathbb{E}_x)$ associated to the Hermitian metric K_x. The key claim we will prove is that τ preserves the subgroup G, and so restricts to a conjugation on G. For this, since G is reductive, we know that there is a $\mathbb{C}$-subspace

$$S \leq T^{a,b}\mathbb{E}_x := \mathbb{E}_x^{\otimes a} \otimes \mathbb{E}_x^{*\otimes b}$$

for some a, b such that G is the subgroup of $\mathrm{GL}(\mathbb{E}_x)$ fixing S pointwise. We may as well take S to be the largest such subspace, i.e.

$$S = (T^{a,b}\mathbb{E}_x)^G = (T^{a,b}\mathbb{E}_x)^{\pi_1(X)} = \mathrm{H}^0(X, T^{a,b}\mathbb{E}) ,$$

where $T^{a,b}\mathbb{E} = \mathbb{E}^{\otimes a} \otimes \mathbb{E}^{*\otimes b}$ (which is a $\mathbb{C}$-PVHS in a natural way). The theorem of the fixed part (Theorem 2.10) tells us that S is actually a $\mathbb{C}$-Hodge substructure of $T^{a,b}\mathbb{E}_x$.

Now let $\mathbb{S} = \underline{\mathbb{C}} \otimes_{\mathbb{C}} S$ denote the constant local system associated to S, which is a sub-$\mathbb{C}$-VHS of $T^{a,b}\mathbb{E}$. So if $\mathbb{S}^\perp$ denotes its orthogonal complement with respect to the polarisation, then we have (Lemma 2.6) an orthogonal decomposition

$$T^{a,b}\mathbb{E} = \mathbb{S} \oplus \mathbb{S}^\perp$$

of $\mathbb{C}$-PVHS.

Now any element $g \in \mathrm{GL}(\mathbb{E}_x)(\mathbb{C})$ can be written uniquely as $u\exp(y)$ where $u \in U(\mathbb{E}_x, K_x)(\mathbb{R})$ and $y \in \mathrm{Lie}(\mathrm{GL}(\mathbb{E}_x)) = \mathrm{End}(\mathbb{E}_x)$ is in the -1 eigenspace for τ, i.e. y is Hermitian symmetric with respect to K_x. If we let $\rho'\colon \mathrm{GL}(\mathbb{E}_x) \to \mathrm{GL}(T^{a,b}\mathbb{E}_x)$ denote the natural representation, then $\rho'(u)$ is a unitary automorphism (with respect to the induced Hermitian metric on $T^{a,b}\mathbb{E}_x$) and $\mathrm{d}\rho'(y)$ is a Hermitian-antisymmetric endomorphism. If the element g lies in $G(\mathbb{C})$, then by definition $\rho'(g)$ fixes the subspace S pointwise, and also fixes the orthogonal complement $S^\perp$ setwise since this is the fibre of the local system $\mathbb{S}^\perp$. This implies that the unitary component $\rho'(u)$ of $\rho'(g)$ and $\exp(\mathrm{d}\rho'(y))$ both preserve the decomposition $T^{a,b}\mathbb{E}_x = S \oplus S^\perp$ and act trivially on S. This implies (by choice of S) that $u \in G(\mathbb{C})$ and $y \in \mathrm{Lie}(G)$. In particular, we have

$$\tau(g) = u\exp(-y) \in G(\mathbb{C})\,,$$

so τ restricts to a conjugation on G. The associated real form is $U = \mathrm{Res}^{\mathbb{C}}_{\mathbb{R}}\, G^\tau$.

It is clear that $U(\mathbb{R})$ is compact. Finally, it is clear that $U(\mathbb{R})$ meets every component of $G(\mathbb{C})$, since for any element $g = u\exp(y) \in G(\mathbb{C})$, the path $[0,1] \to G(\mathbb{C})$ given by

$$t \mapsto u\exp(ty)$$

connects $u \in U(\mathbb{R})$ to $g \in G(\mathbb{C})$. □

Having shown that U is a real form of G, we can now show the same for the real monodromy group W.

Lemma 3.5. *W is a real form of G.*

Proof. Let $\psi\colon \mathbb{G}_m \to \mathrm{GL}(\mathbb{E}_x)$ be the action of $\mathbb{G}_m$ on $\mathbb{E}_x$ where λ acts on $\mathbb{E}_x^{p,q}$ by multiplication by λ^p. Since the space $S \leq T^{a,b}\mathbb{E}_x$ considered above is a $\mathbb{C}$-Hodge substructure, it is stable under the action of $\mathbb{G}_m$. This implies that $\mathbb{G}_m$ normalises the subgroup $G \leq \mathrm{GL}(\mathbb{E}_x)$, and so there is an induced conjugation action of $\mathbb{G}_m$ on G.

By inspection, the unit circle group $U(1) \leq \mathrm{Res}^{\mathbb{C}}_{\mathbb{R}}\,\mathbb{G}_m$ commutes with the conjugation τ (which is the inverse Hermitian adjoint with respect to K_x). Let C be the automorphism of G given by the action of $-1 \in \mathbb{C}^\times$ and let

$$\sigma \coloneqq C\tau = \tau C\,.$$

It follows that σ is another complex conjugation on G; let W' denote the corresponding real form. For any $g \in G(\mathbb{C})$ we have

$$\begin{aligned} Q_x(g\alpha, \sigma(g)\beta) &= Q_x(g\alpha, \psi(-1)\cdot\tau(g)(\psi(-1)\cdot\beta)) \\ &= i^k K_x(g\alpha, \tau(g)(\psi(-1)\cdot\beta)) \\ &= i^k K_x(\alpha, \psi(-1)\cdot\beta) \\ &= Q_x(\alpha,\beta)\,, \end{aligned}$$

and so σ is the inverse Hermitian adjoint with respect to the indefinite sesquilinear pairing Q_x. So W' is the subgroup of G preserving Q_x.

Since the polarisation Q on $\mathbb{E}$ is a morphism of local systems, it follows that the action of $\pi_1(X)$ on $\mathbb{E}_x$ preserves Q_x, so $W \leq W'$. In fact, we must have equality since ρ is Zariski-dense in both W and $G = W'_{\mathbb{C}}$. □

Let us summarise what we have learned in a definition.

Definition 3.6. A real affine algebraic group W is said to be of *Hodge type* just when there is an action of $\mathbb{G}_m$ on the complexification $G = W_{\mathbb{C}}$ such that:

- the action of $U(1)$ commutes with the conjugation σ associated to W (equivalently, $U(1)$ preserves W setwise); and
- if C denotes the action of $-1 \in \mathbb{C}^\times$, then the real form associated to the conjugation
$$\tau = C\sigma = \sigma C$$
is a compact real form of G.

Theorem 3.7. *The real monodromy group W associated to a $\mathbb{C}$-PVHS $\mathbb{E}$ on a compact Kähler manifold X is of Hodge type, and the complexification G of W is the complex monodromy group.*

Proof. We've just proved this. □

It turns out that being of Hodge type imposes strong restrictions on the group W, see [1, §4]. For our purposes, we note only that

Lemma 3.8. *For $n \geq 3$, the group $\mathrm{SL}_{n,\mathbb{R}}$ is not of Hodge type.*

4. Rigid representations

Now we begin to return to the study of Kähler groups. One key insight in the work of Simpson is that among representations of fundamental groups of compact Kähler manifolds, a special role is played by those which are *rigid*, which means that they cannot be continuously deformed to a non-isomorphic representation.

Definition 4.1. Let Γ be a finitely presented group and G a complex reductive group. Let $\mathrm{Hom}(\Gamma, G)$ denote the complex variety parametrising homomorphisms $\rho\colon \Gamma \to G(\mathbb{C})$. (If $g_1, \ldots, g_n$ are generators of Γ, then we can identify $\mathrm{Hom}(\Gamma, G)$ as a closed subvariety of G^n, corresponding to seeing where these generators are mapped to in G.) There is an action of G on $\mathrm{Hom}(\Gamma, G)$ by conjugation.

A representation $\rho\colon \Gamma \to G(\mathbb{C})$ is called *rigid* just when its G-orbit in $\mathrm{Hom}(\Gamma, G(\mathbb{C}))$ is open (in the complex topology). This means that any ρ' sufficiently close to ρ must be conjugate to it under an element of G.

Mildly more generally, a representation ρ is called *properly rigid* just when it is rigid as a representation into the Zariski-closure of $\mathrm{im}(\rho)$. (In practice, there is no need to distinguish between rigid and properly rigid representations, since one can usually just shrink the group G.)

The main result we want to extract from Simpson's non-abelian Hodge correspondence asserts that rigid representations of fundamental groups come from complex polarised variations of Hodge structure.

Theorem 4.2. *Let X be a compact Kähler manifold and G a complex reductive group. If a representation $\rho\colon \pi_1(X) \to G(\mathbb{C})$ is (properly) rigid, then the associated local system $\mathbb{E}$ underlies a $\mathbb{C}$-PVHS.*

4.1. The $\mathbb{C}^\times$ action.

We've basically already seen the proof of Theorem 4.2 in previous talks, but I'll recall the method here. There is a natural action of $\mathbb{C}^\times$ on the set of isomorphism classes of Higgs bundles, given by
$$t\colon (E, \theta) \mapsto (E, t\theta)\,.$$

This action lifts to an action on the set of isomorphism classes of *framed* Higgs bundles, i.e. Higgs bundles (E,θ) together with a chosen basis of E_x for $x \in X$ some fixed basepoint. This action preserves the subset of polystable Higgs bundles with vanishing Chern classes, so induces through the non-abelian Hodge correspondence an action of $\mathbb{C}^\times$ on the set of isomorphism classes of semisimple (framed) $\mathbb{C}$-local systems, denoted by

$$t\colon \mathbb{E} \mapsto \mathbb{E}_t \,.$$

Lemma 4.3. *If a semisimple local system $\mathbb{E}$ on a compact Kähler manifold satisfies $\mathbb{E} \cong \mathbb{E}_t$ for some $t \in \mathbb{C}^\times$ not a root of unity, then $\mathbb{E}$ underlies a $\mathbb{C}$-PVHS.*

Proof. Let (E,θ) be the Higgs bundle corresponding to $\mathbb{E}$ under the non-abelian Hodge correspondence. By assumption, there is an isomorphism $f\colon (E,\theta) \xrightarrow{\sim} (E,t\theta)$, i.e. a holomorphic automorphism $f\colon E \to E$ such that $f\theta = t\theta f$. The coefficients of the characteristic polynomial of the automorphism f are then holomorphic functions on X, so constant. In particular, f has the same eigenvalues on every fibre, and we can decompose

$$E = \bigoplus_\lambda E_\lambda$$

according to the generalised eigenvalues λ of f. Note that $\lambda = 0$ is not an eigenvalue of f, since f is an automorphism. Moreover, since

$$(f - t\lambda)^n\theta = t^n\theta(f-\lambda)^n \,,$$

it follows that θ maps E_λ to $E_{t\lambda}\otimes_{\mathcal{O}_X}\Omega^1_X$. Since t is not a root of unity, by grouping up the eigenvalues of f we can decompose E into holomorphic subbundles

$$E = \bigoplus_i E_i$$

indexed by integers i such that θ maps E_i into $E_{i-1}\otimes_{\mathcal{O}_X}\Omega^1_X$. (This says that E has the structure of a *system of Hodge bundles* in the language of Simpson's paper.)

Thus one obtains an $\mathcal{A}$-linear decomposition

$$\mathcal{E} = \bigoplus_i \mathcal{E}_i \tag{2}$$

of the associated flat bundle $\mathcal{E} = \mathcal{A}\otimes_{\mathcal{O}_X} E$. The connection D on $\mathcal{E}$ decomposes as

$$D = \partial_K + \bar\partial + \theta + \bar\theta_K \,,$$

where $\bar\partial$ and θ are induced from the same-named operators on $\mathcal{A}$ and E, and ∂_K and $\bar\theta_K$ are obtained from them via the compatibility conditions

$$K(\bar\partial\alpha,\beta) + K(\alpha,\partial_K\beta) = \bar\partial K(\alpha,\beta) \quad \text{and} \quad K(\theta\alpha,\beta) = K(\alpha,\bar\theta_K\beta)$$

with respect to the harmonic metric K. In particular, $\bar\partial$ and θ are graded of degree 0 and -1 with respect to the decomposition (2).

Now there is a natural action of $U(1)$ on E by automorphisms of Higgs bundles ($\lambda \in U(1)$ acting as λ^i on E_i), and Simpson's construction of the metric K shows that it can be taken to be $U(1)$-invariant, i.e. such that the decomposition (2) is orthogonal. The compatibility conditions then imply that ∂_K and $\bar\theta_K$ are graded of degree 0 and 1, and hence the local system associated to $\mathcal{E}$ is a $\mathbb{C}$-PVHS. □

The other ingredient we need for the proof of Theorem 4.2 is the fact that the $\mathbb{C}^\times$ on local systems preserves monodromy groups.

Lemma 4.4. *Let $\mathbb{E}$ be a semisimple local system. Then for any $t \in \mathbb{C}^\times$, the monodromy groups of $\mathbb{E}$ and $\mathbb{E}_t$ are equal. (The fibres of $\mathbb{E}$ and $\mathbb{E}_t$ can both be identified with E_x where (E, θ) is the Higgs bundle associated to $\mathbb{E}$, so the monodromy groups live inside the same space.)*

Proof. The monodromy group of $\mathbb{E}$ is exactly the group of automorphisms of $\mathbb{E}_x$ which preserve $\mathbb{S}_x \subseteq T^{a,b}\mathbb{E}_x$ setwise for every local subsystem $\mathbb{S} \subseteq T^{a,b}\mathbb{E}$ and every a, b. The non-abelian Hodge correspondence implies that this group can be characterised in terms of the associated Higgs bundle (E, θ): it is the group of automorphisms of E_x which preserve S_x setwise for every degree 0 Higgs subbundle $S \subseteq T^{a,b}E$ and every a, b. Since the tensor powers of (E, θ) and $(E, t\theta)$ have the same Higgs subbundles, this implies the result. $\square$

Consequently, we can prove Theorem 4.2. We know that the representations ρ_t attached to the local systems $\mathbb{E}_t$ all have image contained in G, and moreover $\rho_t \to \rho$ as $t \to 1$ by [1, Lemma 2.8]. So by rigidity, we deduce that ρ_t and ρ are conjugate when t is sufficiently close to 1. So $\mathbb{E}_t$ and $\mathbb{E}$ are isomorphic, and hence (taking t not a root of unity) we get that $\mathbb{E}$ underlies a $\mathbb{C}$-PVHS.

4.2. **Application to Kähler groups.** Finally, we can distil all of this theory into a purely group-theoretic necessary criterion for a group to be Kähler.

Theorem 4.5. *Let Γ be a Kähler group and G a complex reductive group. Then for every (properly) rigid representation $\rho\colon \Gamma \to G(\mathbb{C})$, the real Zariski-closure W of $\mathrm{im}(\rho)$ is of Hodge type, and $G = W_\mathbb{C}$ is its complexification.*

Proof. Combine Theorems 3.7 and 4.2. $\square$

Lemma 4.6. *For $n \geq 3$, the standard representation*

$$\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{C})$$

is rigid.

Consequently, we see that $\mathrm{SL}_n(\mathbb{Z})$ is not Kähler for $n \geq 3$. Indeed, if it were, then $\mathrm{SL}_{n,\mathbb{R}}$ would be of Hodge type by Theorem 4.5, but it is not (Lemma 3.8). $\square$

References

[1] C. Simpson, *Higgs bundles and local systems.*