Non-abelian Bloch–Kato Selmer sets

and an application to heights on abelian varieties

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Introduction

Motivation: an anabelian theory of heights

Anabelian geometry is, loosely speaking, the attempt to study aspects of Diophantine geometry using fundamental groups.

The fundamental groups in question are typically highly structured objects, for instance if X is a connected variety over a field F and $x \in X(F)$ is a rational basepoint, then the étale fundamental group $\pi_1^{\text{ét}}(X;x)$ is a profinite group endowed with a continuous action of the absolute Galois group G_F.

Other types of fundamental group can also be used, e.g. $\mathbb{Q}_p$-pro-unipotent, de Rham or (log-)crystalline fundamental groups, which carry other kinds of structure.

The principal method of studying rational points is via various *non-abelian Kummer maps*, for example the map

$$X(F) \to \mathrm{H}^1(G_F, \pi_1^{\text{ét}}(X;x))$$

assigning to an F-rational point $y \in X(F)$ the class of the *étale torsor of paths* $\pi_1^{\text{ét}}(X;x,y)$.

When F is a number field and X a curve of genus ≥ 2, one of the implications of Grothendieck's *section conjecture* is that the étale Kummer map should be bijective—it is known to be injective.

Even in the absence of the section conjecture, interesting results can be obtained from consideration of non-abelian Kummer maps, for instance M. Kim's anabelian proof of Siegel's theorem (using the $\mathbb{Q}_p$-pro-unipotent fundamental group).

Motivating question (preliminary version)

Let A/F be an abelian variety over a number field, L/A a line bundle and $y \in A(F)$ a rational point. Can we recover the canonical height $\hat{h}_L(y)$ from anabelian data associated to L? Can we do so *explicitly*?

As the canonical height decomposes as a sum of local components, it makes sense to turn this into a purely local question.

Notation

Fix (for the rest of the talk) a prime p, a finite extension $K/\mathbb{Q}_p$, and an algebraic closure $\overline{K}/K$, determining an absolute Galois group G_K.

Recall that for any divisor D on an abelian variety A/K, there is a *Néron function* $A(K) \setminus D(K) \to \mathbb{R}$, which is the unique (up to scaling) function satisfying a certain list of properties. It can be normalised to take values in $\mathbb{Q}$, and is used as the local component of height functions.

Equivalently, for any line bundle L/A, there is a *Néron log-metric* $L^\times(K) = (L \setminus \{0\})(K) \to \mathbb{R}$, which again is uniquely (up to additive constants) determined by a certain list of properties.

Motivating question (definitive version)

Let A/K be an abelian variety, L/A a line bundle and $U/\mathbb{Q}_p$ the $\mathbb{Q}_p$-unipotent fundamental group of $L^\times = L \setminus \{0\}$. Can we recover the Néron log-metric $L^\times(K) \to \mathbb{Q}$ from the non-abelian Kummer map

$$L^\times(K) \to \mathrm{H}^1(G_K, U(\mathbb{Q}_p))?$$

Can we do so *explicitly*?

An example result

An example of the sort of result we seek (for the $\mathbb{Q}_\ell$-unipotent fundamental group) already appears in existing work.

Theorem (Balakrishnan, Dan-Cohen, Kim, Wewers. 2014)

Let X/K be the complement of 0 in an elliptic curve E/K, and U_2 the 2-step $\mathbb{Q}_\ell$-unipotent fundamental group ($\ell \neq p$) of X. Then the natural map $\mathbb{Q}_\ell(1) \to U_2$ induces a bijection on H^1, and the composite map

$$X(K) \to \mathrm{H}^1(G_K, U_2(\mathbb{Q}_\ell)) \overset{\sim}{\leftarrow} \mathrm{H}^1(G_K, \mathbb{Q}_\ell(1)) \overset{\sim}{\to} \mathbb{Q}_\ell$$

is a $\mathbb{Q}$-valued Néron function on E with divisor [0], *postcomposed with the natural embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_\ell$.*

Local (abelian) Bloch–Kato Selmer groups

- S. Bloch and K. Kato define, for any de Rham representation V of G_K on a $\mathbb{Q}_p$-vector space, subspaces

$$\mathrm{H}^1_e(G_K, V) \leq \mathrm{H}^1_f(G_K, V) \leq \mathrm{H}^1_g(G_K, V)$$

  of the Galois cohomology $\mathrm{H}^1(G_K, V)$.
- Their dimensions are easily computable, and $\mathrm{H}^1_e(G_K, V)$ can be studied via an "exponential" exact sequence

$$0 \to V^{G_K} \to \mathsf{D}^{\varphi=1}_{\mathrm{cris}}(V) \to \mathsf{D}_{\mathrm{dR}}(V)/\mathsf{D}^+_{\mathrm{dR}}(V) \to \mathrm{H}^1_e(G_K, V) \to 0.$$

- When $V = V_pA$ is the $\mathbb{Q}_p$ Tate module of an abelian variety A/K, these are all equal to the $\mathbb{Q}_p$-span of the image of the Kummer map

$$A(K) \to \mathrm{H}^1(G_K, V_pA).$$

Local non-abelian Bloch–Kato Selmer sets

- M. Kim defines, for a unipotent group $U/\mathbb{Q}_p$ with G_K-action (satisfying certain conditions), a pointed subset $\mathrm{H}^1_f(G_K, U(\mathbb{Q}_p))$ of $\mathrm{H}^1(G_K, U(\mathbb{Q}_p))$, which is even the $\mathbb{Q}_p$-points of a scheme $\mathrm{H}^1_f(G_K, U)/\mathbb{Q}_p$.
- It can be studied via an "exponential" isomorphism

$$\mathsf{D}^+_{\mathrm{dR}}(U)\backslash\mathsf{D}_{\mathrm{dR}}(U) \xrightarrow{\sim} \mathrm{H}^1_f(G_K, U).$$

- When $U = U_n$ is the n-step $\mathbb{Q}_p$-unipotent fundamental group of $\mathbb{P}^1_K \setminus \{0, 1, \infty\}$, $\mathrm{H}^1_f(G_K, U(\mathbb{Q}_p))$ is the Zariski closure of the image of the non-abelian Kummer map

$$\mathbb{P}^1 \setminus \{0, 1, \infty\}(\mathcal{O}_K) \to \mathrm{H}^1(G_K, U_n(\mathbb{Q}_p)).$$

Content of this talk

- In this talk we will recall the definition, for a de Rham representation of G_K on a unipotent group $U/\mathbb{Q}_p$, of pointed subsets $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) \subseteq \mathrm{H}^1_f(G_K, U(\mathbb{Q}_p)) \subseteq \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$ of $\mathrm{H}^1(G_K, U(\mathbb{Q}_p))$.
- We will also make sense of the relative quotients of the local Bloch–Kato Selmer sets, e.g. $\mathrm{H}^1_{g/e} = \mathrm{H}^1_g/\mathrm{H}^1_e$.
- We will develop homotopical-algebraic techniques for studying these local Bloch–Kato Selmer sets. These will, for instance, provide us with an "exponential" exact sequence for $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p))$, and give related results for $\mathrm{H}^1_f(G_K, U(\mathbb{Q}_p))$ and $\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$.

The main theorem

Assuming suitably general comparison theorems for fundamental groupoids, we can completely answer the motivating question.

Theorem (B.)

Let A/K be an abelian variety, $L^\times/A$ the complement of zero in a line bundle L, and let U be the $\mathbb{Q}_p$-unipotent fundamental group of $L^\times$. Then U is de Rham, the natural map $\mathbb{Q}_p(1) \to U$ induces a bijection on $\mathrm{H}^1_{g/e}$, and the composite map

$$L^\times(K) \to \mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) \overset{\sim}{\leftarrow} \mathrm{H}^1_{g/e}(G_K, \mathbb{Q}_p(1)) \overset{\sim}{\to} \mathbb{Q}_p$$

is (well-defined and) the Néron log-metric on L.

Proof later

Archimedean analogue

Theorem (B.)

Let $A/\mathbb{C}$ be an abelian variety, $L^\times/A$ the complement of zero in a line bundle L, and let $U = \mathbb{R} \otimes \pi_1(L^\times(\mathbb{C}))$ be the $\mathbb{R}$-unipotent fundamental group of $L^\times$, endowed with its $\mathbb{R}$-mixed Hodge structure. Then the natural map $\mathbb{R}(1) \to U$ induces a bijection on H^1, and the composite map

$$L^\times(\mathbb{C}) \to \mathrm{H}^1(U) \xleftarrow{\sim} \mathrm{H}^1(\mathbb{R}(1)) \xrightarrow{\sim} \mathbb{R}$$

is the Néron log-metric on L.
Here $\mathrm{H}^1(U)$ denotes the set of isomorphism classes of U-torsors with compatible $\mathbb{R}$-mixed Hodge structure.

Basic concepts

Galois representations on unipotent groups

Definition (Galois representations on unipotent groups)

A *representation of G_K on a unipotent group $U/\mathbb{Q}_p$* is an action of G_K on U (by algebraic automorphisms) such that the action on $U(\mathbb{Q}_p)$ is continuous.
We say that U is *de Rham* (resp. semistable, crystalline etc.) just when the following equivalent conditions hold:

- $\mathrm{Lie}(U)$ is de Rham;
- $\mathcal{O}(U)$ is ind-de Rham;
- $\dim_K(\mathsf{D}_{\mathrm{dR}}(U)) = \dim_{\mathbb{Q}_p}(U)$, where $\mathsf{D}_{\mathrm{dR}}(U)/K$ is the unipotent group representing the functor

$$\mathsf{D}_{\mathrm{dR}}(U)(A) := U(A \otimes_K \mathsf{B}_{\mathrm{dR}})^{G_K}.$$

Definition (Local non-abelian Bloch–Kato Selmer sets)

Let $U/\mathbb{Q}_p$ be a de Rham representation of G_K on a unipotent group. We define pointed subsets

$$\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) \subseteq \mathrm{H}^1_f(G_K, U(\mathbb{Q}_p)) \subseteq \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))$$

of the non-abelian cohomology $\mathrm{H}^1(G_K, U(\mathbb{Q}_p))$ to be the kernels

$$\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) := \ker\left(\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{cris}}^{\varphi=1}))\right);$$
$$\mathrm{H}^1_f(G_K, U(\mathbb{Q}_p)) := \ker\left(\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{cris}}))\right);$$
$$\mathrm{H}^1_g(G_K, U(\mathbb{Q}_p)) := \ker\left(\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{st}}))\right).$$

One can use B_{dR} in place of B_{st} in the definition of H^1_g.

Definition (Quotients of Bloch–Kato Selmer sets)

Let $U/\mathbb{Q}_p$ be a de Rham representation of G_K on a unipotent group. We denote by $\sim_{\mathrm{H}^1_e}$, $\sim_{\mathrm{H}^1_f}$, $\sim_{\mathrm{H}^1_g}$ the equivalence relations on $\mathrm{H}^1(G_K, U(\mathbb{Q}_p))$ whose equivalence classes are the fibres of

$$\begin{aligned}
\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) &\to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{cris}}^{\varphi=1}));\\
\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) &\to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{cris}}));\\
\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) &\to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{st}})).
\end{aligned}$$

We then define, for instance, the *Bloch–Kato quotient*

$$\mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) := \mathrm{H}^1_g(G_K, U(\mathbb{Q}_p))/\sim_{\mathrm{H}^1_e}.$$

Why a cosimplicial approach?

The abelian Bloch–Kato exponential for a de Rham representation V arises from tensoring it with the exact sequence

$$0 \to \mathbb{Q}_p \to \mathsf{B}_{\mathrm{cris}}^{\varphi=1} \to \mathsf{B}_{\mathrm{dR}}/\mathsf{B}_{\mathrm{dR}}^+ \to 0$$

and taking the long exact sequence in Galois cohomology. Equivalently, if we consider the cochain complex

$$\mathsf{C}_e^\bullet : \mathsf{B}_{\mathrm{cris}}^{\varphi=1} \to \mathsf{B}_{\mathrm{dR}}/\mathsf{B}_{\mathrm{dR}}^+$$

(which is a resolution of $\mathbb{Q}_p$), then the cohomology groups of the cochain $(\mathsf{C}_e^\bullet \otimes_{\mathbb{Q}_p} V)^{G_K}$ are canonically identified as

$$\mathrm{H}^j\left((\mathsf{C}_e^\bullet \otimes_{\mathbb{Q}_p} V)^{G_K}\right) \cong \begin{cases} V^{G_K} & j = 0;\\ \mathrm{H}^1_e(G_K, V) & j = 1;\\ 0 & j \geq 2. \end{cases}$$

The advantage of using cochain complexes is that we can perform analogous constructions for H^1_f and H^1_g. For instance, taking the cochain complex

$$\mathsf{C}^\bullet_g : \mathsf{B}_{\mathrm{st}} \to \mathsf{B}_{\mathrm{st}}^{\oplus 2} \oplus \mathsf{B}_{\mathrm{dR}}/\mathsf{B}_{\mathrm{dR}}^+ \to \mathsf{B}_{\mathrm{st}}$$

(which is also a resolution of $\mathbb{Q}_p$), the cohomology groups of $(\mathsf{C}^\bullet_g \otimes_{\mathbb{Q}_p} V)^{G_K}$ are canonically identified as

$$\mathrm{H}^j\left((\mathsf{C}^\bullet_g \otimes_{\mathbb{Q}_p} V)^{G_K}\right) \cong \begin{cases} V^{G_K} & j = 0; \\ \mathrm{H}^1_g(G_K, V) & j = 1; \\ \mathsf{D}_{\mathrm{cris}}^{\varphi=1}(V^*(1))^* & j = 2; \\ 0 & j \geq 3. \end{cases}$$

The cochain complexes $\mathsf{C}^\bullet_e$, $\mathsf{C}^\bullet_f$, $\mathsf{C}^\bullet_g$ themselves cannot be directly be used in the non-abelian setting (as we cannot tensor a group by a vector space), so we have to tweak them slightly to find a non-abelian generalisation of the Bloch–Kato exponential.

For example, in place of $\mathsf{C}^\bullet_e$, we consider the diagram

$$\mathsf{B}_{\mathrm{cris}}^{\varphi=1} \times \mathsf{B}_{\mathrm{dR}}^+ \rightrightarrows \mathsf{B}_{\mathrm{dR}}$$

of $\mathbb{Q}_p$-algebras. Taking points in U and then G_K-fixed points, we then obtain the diagram

$$\mathsf{D}_{\mathrm{cris}}^{\varphi=1}(U)(K_0) \times \mathsf{D}_{\mathrm{dR}}^+(U)(K) \rightrightarrows \mathsf{D}_{\mathrm{dR}}(U)(K).$$

By considering the action of $\mathsf{D}_{\mathrm{cris}}^{\varphi=1}(U)(K_0) \times \mathsf{D}_{\mathrm{dR}}^+(U)(K)$ on $\mathsf{D}_{\mathrm{dR}}(U)(K)$ by $(x, y)\colon z \mapsto y^{-1}zx$, we arrive at a non-abelian analogue of the Bloch–Kato exponential.

The non-abelian Bloch–Kato exponential (explicit version)

Theorem (B.)

Let $U/\mathbb{Q}_p$ be a de Rham representation of G_K on a unipotent group. Then the action of $\mathsf{D}_{\mathrm{cris}}^{\varphi=1}(U)(K_0) \times \mathsf{D}_{\mathrm{dR}}^{+}(U)(K)$ on $\mathsf{D}_{\mathrm{dR}}(U)(K)$ by $(x, y)\colon z \mapsto y^{-1}zx$ has orbit space $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p))$ and point-stabiliser $U(\mathbb{Q}_p)^{G_K}$.

In particular, we canonically identify $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p))$ with the double-coset space

$$\mathsf{D}_{\mathrm{dR}}^{+}(U)(K)\backslash \mathsf{D}_{\mathrm{dR}}(U)(K)/\mathsf{D}_{\mathrm{cris}}^{\varphi=1}(U)(K_0).$$

Proof later

Remark

When $\mathsf{D}_{\mathrm{cris}}^{\varphi=1}(U) = 1$ (as in Kim's and Sakugawa's work), we obtain

$$\mathrm{H}^1_f(G_K, U(\mathbb{Q}_p)) = \mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) \cong \mathsf{D}_{\mathrm{dR}}^{+}(U)(K)\backslash \mathsf{D}_{\mathrm{dR}}(U)(K),$$

which recovers their descriptions of $\mathrm{H}^1_f(G_K, U(\mathbb{Q}_p))$.

Non-abelian analogy

In order to extend the study of local Bloch–Kato Selmer groups to the non-abelian context, we need to replace three abelian concepts with non-abelian analogues.

- In place of the cochain complexes $\mathrm{C}^\bullet_*$ of G_K-representations, we will use *cosimplicial* $\mathbb{Q}_p$*-algebras* $\mathrm{B}^\bullet_*$ with G_K-action.
- In place of the cochain complexes $(\mathrm{C}^\bullet_* \otimes_{\mathbb{Q}_p} V)^{G_K}$, we will examine the *cosimplicial groups* $U(\mathrm{B}^\bullet_*)^{G_K}$.
- In place of the cohomology groups of these cochain complexes, we will calculate the *cohomotopy groups/sets* of the corresponding cosimplicial groups.

Cosimplicial groups

Definition (Cosimplicial objects)

A *cosimplicial object* of a category $\mathcal{C}$ is a covariant functor $X^\bullet\colon \Delta \to \mathcal{C}$ from the simplex category Δ of non-empty finite ordinals and order-preserving maps. We think of this as a collection of objects X^n together with *coface* maps $d^\bullet$

$$X^0 \rightrightarrows X^1 \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} X^2 \cdots$$

and *codegeneracy* maps $s^\bullet$

$$X^0 \leftarrow X^1 \leftleftarrows X^2 \cdots$$

satisfying certain identities.

Remark

Cosimplicial groups are a non-abelian generalisation of cochain complexes of abelian groups. Specifically, the category of coconnected cochain complexes is equivalent to the category of *abelian* cosimplicial groups (by the cosimplicial Dold–Kan correspondence).

We seek an invariant for cosimplicial groups generalising cohomology of cochain complexes.

Definition (Cohomotopy groups/sets)

Let $U^\bullet$ be a cosimplicial group

$$U^0 \rightrightarrows U^1 \substack{\rightarrow\\\rightarrow\\\rightarrow} U^2 \cdots .$$

We define the 0*th cohomotopy group* $\pi^0(U^\bullet)$ to be

$$\pi^0(U^\bullet) := \{u^0 \in U^0 \mid d^0(u^0) = d^1(u^0)\} \leq U^0.$$

We also define the pointed set of 1-*cocycles* to be

$$\mathrm{Z}^1(U^\bullet) := \{u^1 \in U^1 \mid d^1(u^1) = d^2(u^1)d^0(u^1)\} \subseteq U^1$$

and the 1*st cohomotopy (pointed) set* $\pi^1(U^\bullet) := \mathrm{Z}^1(U^\bullet)/U^0$ to be the quotient of $\mathrm{Z}^1(U^\bullet)$ by the twisted conjugation action of U^0, given by $u^0\colon u^1 \mapsto d^1(u^0)^{-1}u^1d^0(u^0)$.

Definition (Cohomotopy groups/sets (cont.))

When $U^\bullet$ is abelian, $\pi^0(U^\bullet)$ and $\pi^1(U^\bullet)$ are abelian groups, and we can define the higher cohomotopy groups $\pi^j(U^\bullet)$ to be the cohomology groups of the cochain complex

$$U^0 \to U^1 \to U^2 \cdots$$

with differential $\sum_k (-1)^k d^k$.

In this way, cohomotopy of cosimplicial groups generalises cohomology of cochain complexes.

Example (Non-abelian group cohomology)

Suppose G is a topological group acting continuously on another topological group U. Then $\mathrm{C}^n(G, U) := \mathrm{Map}_{\mathrm{cts}}(G^n, U)$ can be given the structure of a cosimplicial group. Its cohomotopy $\pi^j(\mathrm{C}^\bullet(G, U))$ is canonically identified with the group cohomology $\mathrm{H}^j(G, U)$ for $j = 0, 1$, and for all j when U is abelian.

Long exact sequences in cohomotopy

Notation

When we assert that a sequence

$$\cdots \to U^{r-1} \to U^r \overset{\curvearrowright}{\to} U^{r+1} \to U^{r+2} \to \cdots$$

is *exact*, we shall mean that:

- $\cdots \to U^{r-1} \to U^r$ is an exact sequence of groups (and group homomorphisms);
- $U^{r+1} \to U^{r+2} \to \cdots$ is an exact sequence of pointed sets;
- there is an action of U^r on U^{r+1} whose orbits are the fibres of $U^{r+1} \to U^{r+2}$, and whose point-stabiliser is the image of $U^{r-1} \to U^r$.

Cosimplicial groups give us many ways of producing long exact sequences of groups and pointed sets. For example:

Theorem (Bousfield, Kan. 1972)

Let

$$1 \to Z^\bullet \to U^\bullet \to Q^\bullet \to 1$$

be a central extension of cosimplicial groups. Then there is a cohomotopy exact sequence

$$1 \longrightarrow \pi^0(Z^\bullet) \longrightarrow \pi^0(U^\bullet) \longrightarrow \pi^0(Q^\bullet) \longrightarrow \pi^1(Z^\bullet) \overset{\curvearrowright}{\longrightarrow} \pi^1(U^\bullet) \longrightarrow \pi^1(Q^\bullet) \longrightarrow \pi^2(Z^\bullet).$$

Cosimplicial Bloch–Kato theory

Methodology

Our general method for studying local Bloch–Kato Selmer sets and their quotients will be to define various cosimplicial $\mathbb{Q}_p$-algebras $\mathrm{B}_e^\bullet$, $\mathrm{B}_f^\bullet$, $\mathrm{B}_g^\bullet$, $\mathrm{B}_{g/e}^\bullet$, $\mathrm{B}_{f/e}^\bullet$ with G_K-action such that, for any de Rham representation of G_K on a unipotent group $U/\mathbb{Q}_p$, we have a canonical identification

$$\pi^1\left(U(\mathrm{B}_*^\bullet)^{G_K}\right) \cong \mathrm{H}_*^1(G_K, U(\mathbb{Q}_p)).$$

Cohomotopy of the cosimplicial Dieudonné functors

In fact, we can give a complete description of the cohomotopy groups/sets of each $U(\mathrm{B}_*^\bullet)^{G_K}$. For instance, we have

$$\pi^j\left(U(\mathrm{B}_e^\bullet)^{G_K}\right) \cong \begin{cases} U(\mathbb{Q}_p)^{G_K} & j=0; \\ \mathrm{H}^1_e(G_K, U(\mathbb{Q}_p)) & j=1; \\ 0 & j\geq 2 \text{ and } U \text{ abelian;} \end{cases}$$

$$\pi^j\left(U(\mathrm{B}_{g/e}^\bullet)^{G_K}\right) \cong \begin{cases} \mathrm{D}_{\mathrm{cris}}^{\varphi=1}(U)(K_0) & j=0; \\ \mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) & j=1; \\ \mathrm{D}_{\mathrm{cris}}^{\varphi=1}(U(\mathbb{Q}_p)^*(1))^* & j=2 \text{ and } U \text{ abelian;} \\ 0 & j\geq 3 \text{ and } U \text{ abelian.} \end{cases}$$

Construction of Bloch–Kato algebras

The cosimplicial algebras required to make this work are all built from standard period rings. For example, the diagram

$$\mathrm{B}_{\mathrm{cris}}^{\varphi=1}\times \mathrm{B}_{\mathrm{dR}}^+ \rightrightarrows \mathrm{B}_{\mathrm{dR}}$$

(which we saw earlier) is a semi-cosimplicial $\mathbb{Q}_p$-algebra (that is, a cosimplicial algebra without codegeneracy maps). $\mathrm{B}_e^\bullet$ is then the universal cosimplicial $\mathbb{Q}_p$-algebra mapping to this semi-cosimplicial algebra (the cosimplicial algebra cogenerated by it). Concretely, this has terms

$$\mathrm{B}_e^n = \mathrm{B}_{\mathrm{cris}}^{\varphi=1}\times \mathrm{B}_{\mathrm{dR}}^+\times \mathrm{B}_{\mathrm{dR}}^n.$$

The non-abelian Bloch–Kato exponential

The description of the cohomotopy of $U(\mathrm{B}_e^\bullet)^{G_K}$ in degrees 0 and 1 is equivalent to our earlier explicit statement of the non-abelian Bloch–Kato exponential, which is in turn equivalent to the existence of a *non-abelian exponential exact sequence*

$$1 \longrightarrow U(\mathbb{Q}_p)^{G_K} \longrightarrow \mathrm{D}_{\mathrm{cris}}^{\varphi=1}(U)(K_0) \times \mathrm{D}_{\mathrm{dR}}^{+}(U)(K) \overset{\curvearrowright}{\longrightarrow} \mathrm{D}_{\mathrm{dR}}(U)(K) \overset{\exp}{\to} \mathrm{H}_e^1(G_K, U(\mathbb{Q}_p)) \longrightarrow 1.$$

This is what we will prove.

Construction of the non-abelian Bloch–Kato exponential

By induction along the central series of U, we see quickly that $\pi^0\left(U(\mathrm{B}_e^\bullet)\right) = U(\mathbb{Q}_p)$ and $\pi^1\left(U(\mathrm{B}_e^\bullet)\right) = 1$. Unpacking the definition of $\mathrm{B}_e^\bullet$, this says that

$$1 \to U(\mathbb{Q}_p) \to U(\mathrm{B}_{\mathrm{cris}}^{\varphi=1}) \times U(\mathrm{B}_{\mathrm{dR}}^{+}) \overset{\curvearrowright}{\to} U(\mathrm{B}_{\mathrm{dR}}) \to 1$$

is exact (i.e. the action is transitive with point-stabiliser $U(\mathbb{Q}_p)$). We then obtain a long exact sequence in Galois cohomology

$$1 \longrightarrow U(\mathbb{Q}_p)^{G_K} \longrightarrow \mathrm{D}_{\mathrm{cris}}^{\varphi=1}(U)(K_0) \times \mathrm{D}_{\mathrm{dR}}^{+}(U)(K) \overset{\curvearrowright}{\longrightarrow} \mathrm{D}_{\mathrm{dR}}(U)(K) \overset{\exp}{\to} \mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \longrightarrow \mathrm{H}^1(G_K, U(\mathrm{B}_{\mathrm{cris}}^{\varphi=1}) \times U(\mathrm{B}_{\mathrm{dR}}^{+})),$$

which is already most of the desired exponential sequence.

Construction of the non-abelian Bloch–Kato exponential (cont.)

It remains to show that the image of exp is exactly $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p))$. The exact sequence shows that the image is exactly the kernel of

$$\mathrm{H}^1(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{cris}}^{\varphi=1})) \times \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{dR}}^+)),$$

which certainly is contained in $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p))$.

It is then not too hard to prove that in fact the kernel is exactly $\mathrm{H}^1_e(G_K, U(\mathbb{Q}_p))$, using the fact that the map

$$\mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{dR}}^+)) \to \mathrm{H}^1(G_K, U(\mathsf{B}_{\mathrm{dR}}))$$

has trivial kernel (we omit the diagram-chase in the interests of brevity). This establishes the desired exact sequence, and hence the description of the cohomotopy of $U(\mathsf{B}_e^\bullet)^{G_K}$. □

Proof of the main theorem

In order to prove our main theorem, we need a simple preparatory lemma, and a comparison theorem (currently unproven/uncited).

Lemma

Let

$$1 \to Z \to U \to Q \to 1$$

be a central extension of de Rham representations of G_K on unipotent groups over $\mathbb{Q}_p$. Then there is an exact sequence

$$1 \longrightarrow \mathrm{D}_{\mathrm{cris}}^{\varphi=1}(Z)(K_0) \longrightarrow \mathrm{D}_{\mathrm{cris}}^{\varphi=1}(U)(K_0) \longrightarrow \mathrm{D}_{\mathrm{cris}}^{\varphi=1}(Q)(K_0) \to \mathrm{H}^1_{g/e}(G_K, Z(\mathbb{Q}_p)) \curvearrowright \mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1_{g/e}(G_K, Q(\mathbb{Q}_p)) \to \mathrm{D}_{\mathrm{cris}}^{\varphi=1}(Z(\mathbb{Q}_p)^*(1))^*.$$

Proof of lemma.

From the construction of $\mathrm{B}^\bullet_{g/e}$ (out of B_{st}), it follows that

$$1 \to Z(\mathrm{B}^\bullet_{g/e})^{G_K} \to U(\mathrm{B}^\bullet_{g/e})^{G_K} \to Q(\mathrm{B}^\bullet_{g/e})^{G_K} \to 1$$

is a central extension of cosimplicial groups. The desired exact sequence is then the cohomotopy exact sequence for these cosimplicial groups. □

π_1 comparison (conjecture)

Let X be a (semistable) $\mathcal{O}_K$-scheme, endowed with the log structure induced from a normal crossings divisor D containing the special fibre X_s, and suppose that $X \to \mathrm{Spec}(\mathcal{O}_K)$ is proper and log-smooth, where $\mathrm{Spec}(\mathcal{O}_K)$ is endowed with the log structure induced from the special point. Let $x, y \in X(\mathcal{O}_K)$ be sections of $X \to \mathrm{Spec}(\mathcal{O}_K)$ compatible with the log structures.

Then there are isomorphisms

$$\mathsf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} U_2^{\mathbb{Q}_p}(X_\eta; x_\eta, y_\eta) \xrightarrow{\sim} \mathsf{B}_{\mathrm{dR}} \otimes_K U_2^{\mathrm{dR}}(X_\eta; x_\eta, y_\eta)$$

$$\mathsf{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} U_2^{\mathbb{Q}_p}(X_\eta; x_\eta, y_\eta) \xrightarrow{\sim} \mathsf{B}_{\mathrm{st}} \otimes_{K_0} U_2^{\mathrm{cris}}(X_s/K_0; x_s, y_s)$$

relating the $\mathbb{Q}_p$-unipotent, de Rham and log-crystalline path-torsors at depth 2, respecting all structures (Galois action, Hodge filtration, Frobenius, monodromy).

(Conditional) proof of the main theorem cf. earlier statement

It follows (e.g. from comparison with Betti fundamental groups) that U is a central extension

$$1 \to \mathbb{Q}_p(1) \to U \to V_pA \to 1.$$

By the étale-de Rham comparison theorem, U (and V_pA and $\mathbb{Q}_p(1)$) are de Rham, so we have an exact sequence

$$\mathsf{D}_{\mathrm{cris}}^{\varphi=1}(V_pA) \to \mathrm{H}^1_{g/e}(G_K, \mathbb{Q}_p(1)) \xrightarrow{\curvearrowright} \mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) \to \mathrm{H}^1_{g/e}(G_K, V_pA).$$

But the outer terms vanish (e.g. by p-adic weight-monodromy for abelian varieties), so $\mathrm{H}^1_{g/e}(G_K, \mathbb{Q}_p(1)) \to \mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p))$ is bijective.

(Conditional) proof of the main theorem (cont.)

It also follows from the étale-de Rham comparison theorem that the non-abelian Kummer map

$$L^\times(K) \to \mathrm{H}^1(G_K, U(\mathbb{Q}_p))$$

has image contained in H^1_g, and hence that the composite

$$\lambda\colon L^\times(K) \to \mathrm{H}^1_{g/e}(G_K, U(\mathbb{Q}_p)) \xleftarrow{\sim} \mathrm{H}^1_{g/e}(G_K, \mathbb{Q}_p(1)) \xrightarrow{\sim} \mathbb{Q}_p$$

is well-defined.

To show this map is the Néron log-metric, it remains to show that it satisfies a certain list of properties. These are mostly completely formal, with the exception of local constancy, which requires comparison with log-crystalline path-torsors. □

Questions or comments?