Unexpected algebraic points in non-abelian Chabauty loci

ongoing work, incl. joint work with Jennifer Balakrishnan

slides available at lalexanderbetts.net

Aim of this talk

A new conceptualisation of quadratic Chabauty which is of a more algebraic (as opposed to p-adic analytic) nature, and its connections to the theory of unlikely intersections.

Motivation: Kim's Conjecture

Setup and notation

- X a smooth hyperbolic curve over $\mathbb{Q}$, and $b \in X(\mathbb{Q})$
- $g =$ genus, $J = \mathrm{Jac}(X)$, $r = \mathrm{rk}(J(\mathbb{Q}))$, $\rho = \mathrm{rk}(NS(J)(\mathbb{Q}))$.
- p a prime of good reduction; U a quotient of $\pi_1^{\mathbb{Q}_p}(X_{\overline{\mathbb{Q}}}; b)$
  - $U = U_1$ abelianisation.
  - $U = U_n$ maximal depth n quotient.
  - $U = U_{\mathrm{QC}}$ an extension of U_1 by $\mathbb{Q}_p(1)^{\rho-1}$.

$\rightsquigarrow X(\mathbb{Q}_p)_U \subseteq X(\mathbb{Q}_p)$ containing $X(\mathbb{Q})$

Kim's Conjecture

Conjecture (Kim, '12; Balakrishnan–Dan-Cohen–Kim–Wewers '18):
If U is large enough, then
$$X(\mathbb{Q}_p)_U = X(\mathbb{Q}) .$$

When do we expect $X(\mathbb{Q}_p)_U = X(\mathbb{Q})$?

$X(\mathbb{Q}_p)_U$ is the common vanishing locus of some number of Coleman analytic functions.

- If 0 functions, then $X(\mathbb{Q}_p)_U = X(\mathbb{Q}_p)$. ☹
- If 1 function, then $X(\mathbb{Q}_p)_U$ is finite. ☺
- If ≥ 2 functions, then $X(\mathbb{Q}_p)_U$ is overdetermined. ☺☺ ?

<u>Naïve guess:</u>
If $X(\mathbb{Q}_p)_U$ is cut out by ≥ 2 independent Coleman functions, then it is equal to $X(\mathbb{Q})$.

Example #1

(Balakrishnan–Bianchi–Çiperiani–Cantoral-Farfán–Etropolski, '19)

genus 3, rank 1

X is the hyperelliptic curve with equation

$$y^2 = 4x^7 + 9x^6 - 8x^5 - 9x^4 - 16x^3 + 32x^2 + 32x + 8\,.$$

$$X(\mathbb{Q}_7)_1 = \{\infty, (-1, \pm 1), (1, \pm 5)\} \cup \{\text{Weierstrass pts}\} \cup \{(0, \pm 2\sqrt{2})\}$$

cut out by
2 functions

Example #2

(Hashimoto–Morrison, '21)

X is the Picard curve with equation

$$y^3 = x^4 + 2x^3 + 6x^2 + 5x + 2 \, .$$

genus 3, rank 1

$$X(\mathbb{Q}_{11})_1 = \{\infty\} \cup \{(1/2, \sqrt[3]{13/16})\}$$

cut out by
2 functions

Example #3

(Bianchi, '20)

X is the once-punctured elliptic curve with equation

$$y^2 + xy + y = x^3 - x^2 - 91x - 310\,.$$

rank 0

$$X(\mathbb{Z}_5)_2 = \{(5, 2 \pm i)\}$$

cut out by
2 functions

Example #4

(Corwin–Dan-Cohen, '20)

X is the thrice-punctured line over $\mathbb{Z}[\frac{1}{\ell}]$, ℓ an odd prime.

$$X(\mathbb{Z}_p)_{\{\ell\},\,\mathrm{PL},\,n} \ni -1 \quad \text{for all } n.$$

cut out by arbitrarily many functions

Stoll's Conjecture

Conjecture (Stoll, '07):

Suppose that X is smooth projective, with $g - r \geq 2$. Then there exists a finite subscheme $Z \subset X$ defined over $\mathbb{Q}$ such that

$$X(\mathbb{Q}_p)_1 \subseteq Z(\mathbb{Q}_p)$$

for all primes p in a set of density 1.

⇒ every element of $X(\mathbb{Q}_p)_1$ is algebraic over $\mathbb{Q}$.

⇒ the size of $X(\mathbb{Q}_p)_1$ is bounded uniformly.

Non-abelian Stoll's Conjecture

Conjecture (after Stoll, Bianchi):

Suppose that $U = U^p$ is a quotient of $\pi_1^{\mathbb{Q}_p}(X_{\overline{\mathbb{Q}}}; b)$ of motivic origin, such that $X(\mathbb{Q}_p)_{U^p}$ is cut out by ≥ 2 independent functions. Let U^q be the corresponding quotient of $\pi_1^{\mathbb{Q}_q}(X_{\overline{\mathbb{Q}}}; b)$.

Then there is a finite subscheme $Z \subset X$ defined over $\mathbb{Q}$ such that

$$X(\mathbb{Q}_q)_{U^q} \subseteq Z(\mathbb{Q}_q)$$

for a density 1 set of primes q.

Unlikely Intersections

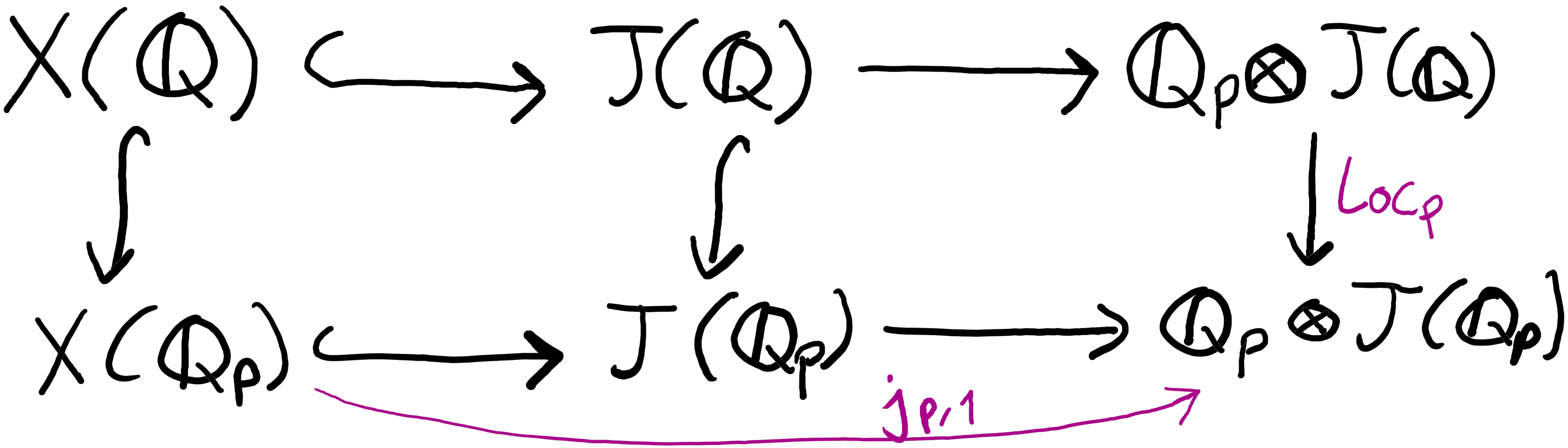
part 1: abelian Chabauty

Abelian Chabauty

Abelian Chabauty, or Chabauty–Coleman, is the special case of non-abelian Chabauty when $U = U_1$ is the abelianisation of $\pi_1^{\mathbb{Q}_p}(X_{\overline{\mathbb{Q}}}; b)$.

The quotient U_1 can also be thought of as the fundamental group of the Jacobian J, and it is possible to describe $X(\mathbb{Q}_p)_1$ purely in terms of J.

Abelian Chabauty



Fact: $X(\mathbb{Q}_p)_1 = j_{p,1}^{-1}\left(\mathrm{im}(loc_p)\right)$

Torsion packets

If $r = 0$, then $X(\mathbb{Q}_p)_1$ is the kernel of the composition

$$X(\mathbb{Q}_p) \longrightarrow J(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p \otimes J(\mathbb{Q}_p)$$

a.k.a. the torsion packet containing $b \in X(\mathbb{Q})$.

Manin–Mumford ⇒ all torsion packets are finite.

Corollary:
Stoll's Conjecture holds when $r = 0$.

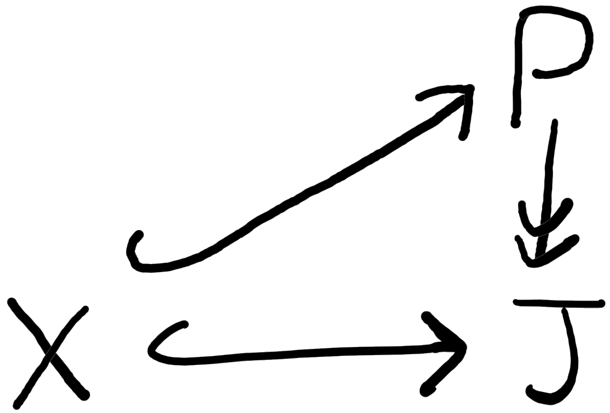
Unlikely Intersections

part 2: quadratic Chabauty

Quadratic Chabauty

Quadratic Chabauty is non-abelian Chabauty for $U = U_{\mathrm{QC}}$ an extension of U_1 by $\mathbb{Q}_p(1)^{\rho-1}$. Which one?

(≈ Edixhoven–Lido, '21) There is a natural $\mathbb{G}_m^{\rho-1}$-torsor P over J whose pullback to X is trivial.



U_{QC} is the fundamental group of P.

Quadratic Chabauty

We give a new description of $X(\mathbb{Q}_p)_{\mathrm{QC}}$ solely in terms of P, rather than the quotient U_{QC}.

- Inspired by the *geometric quadratic Chabauty* obstruction of Edixhoven and Lido.
- Geometric quadratic Chabauty locus $X(\mathbb{Q}_p)_{\mathrm{GQC}} \subseteq X(\mathbb{Q}_p)$ containing $X(\mathbb{Q})$, defined solely in terms of P.
- Geometric quadratic Chabauty is at least as strong as usual quadratic Chabauty, and sometimes stronger (Duque-Rosero–Hashimoto–Spelier, '23).

Cubical structures (Breen, '83)

If A and G are abelian groups, and P is a G-torsor over A, then we define

$$\Theta_3(P) \coloneqq \bigotimes_{I \subseteq \{1,2,3\}} m_I^* P^{(-1)^{3-\#I}} ,$$

where $m_I : A^3 \to A$ is addition of the coordinates in I.

A *cubical structure* on P is a trivialisation of $\Theta_3(P)$ satisfying certain conditions. A *cubical torsor* is a torsor with a cubical structure.

Examples of cubical torsors

- Any $\mathbb{G}_m^r$-torsor over any abelian variety comes with a unique cubical structure (theorem of the cube).
- If P is a central extension of A by G, one can use the group law to endow P with a cubical structure.
- If P is a finitely generated cubical torsor, then it has a *vectorial hull* $\mathbb{Q} \otimes P$, which is a cubical $(\mathbb{Q} \otimes G)$-torsor over $\mathbb{Q} \otimes A$.
- If P is a finitely generated profinite cubical torsor, then it has a *vectorial hull* $\mathbb{Q}_p \otimes P$, which is a cubical $(\mathbb{Q}_p \otimes G)$-torsor over $\mathbb{Q}_p \otimes A$.

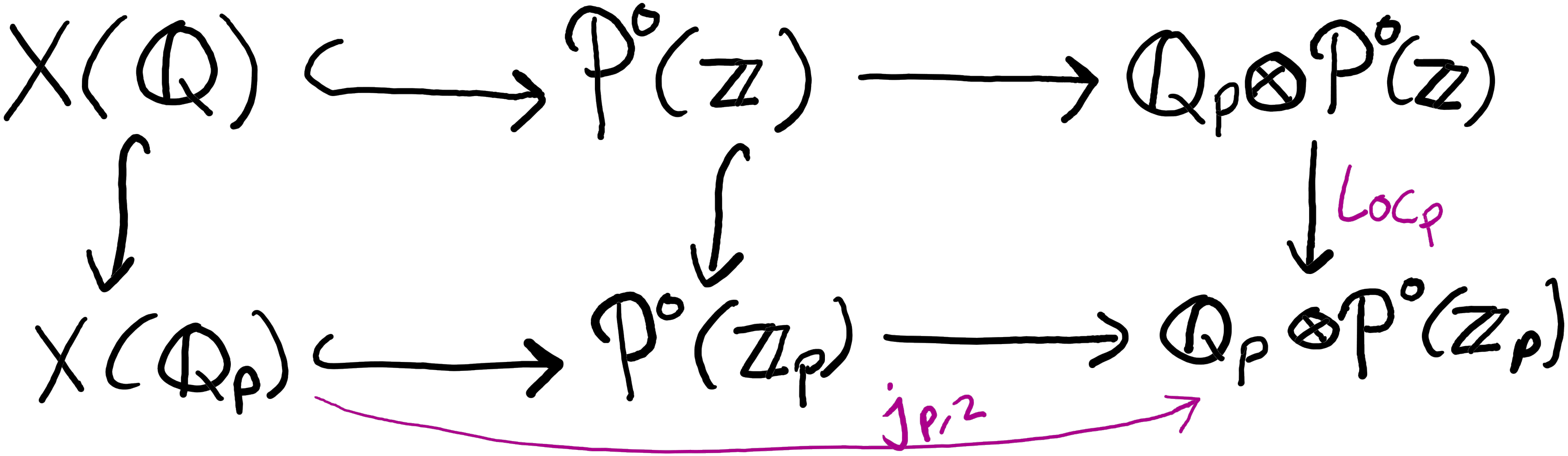
Quadratic Chabauty, revisited

Assume for simplicity that X has everywhere potentially good reduction.

Let $\mathcal{P}^0$ be the identity component of the Neron model of P.

It is a cubical $\mathbb{G}_{m,\mathbb{Z}}^{\rho-1}$-torsor over $\mathcal{J}^0$.

Quadratic Chabauty, revisited



Theorem (B., in progress):

$$X(\mathbb{Q}_p)_{\mathrm{QC}} = j_{p,2}^{-1}\left(\mathrm{im}\left(loc_p\right)\right)$$

Idea of the proof

Lemma (B.):

$$\mathrm{H}^1_f\left(G_p, U_{\mathrm{QC}}\right) \cong \mathbb{Q}_p \otimes \mathcal{P}^0(\mathbb{Z}_p)$$

Proof sketch:

1. The cubical structure on $\mathcal{P}^0$ induces one on $\mathrm{H}^1_f(G_p, U_{\mathrm{QC}})$.
2. The Kummer map $\mathcal{P}^0\left(\mathbb{Z}_p\right) \to \mathrm{H}^1_f(G_p, U_{\mathrm{QC}})$ factors uniquely through a map $\mathbb{Q}_p \otimes \mathcal{P}^0\left(\mathbb{Z}_p\right) \to \mathrm{H}^1_f(G_p, U_{\mathrm{QC}})$.
3. The factored map is an iso because $\mathbb{Q}_p \otimes J\left(\mathbb{Q}_p\right) \to \mathrm{H}^1_f(G_p, U_1)$ and $\mathbb{Q}_p \otimes \mathbb{G}_m\left(\mathbb{Z}_p\right) \to \mathrm{H}^1_f(G_p, \mathbb{Q}_p(1))$ are.

Higher torsion packets

When $r = 0$, we have $\mathbb{Q}_p \otimes \mathcal{P}^0(\mathbb{Z}) = 0$, and so $X(\mathbb{Q}_p)_{\mathrm{QC}}$ is the kernel of the composition

$$X(\mathbb{Q}_p) \longrightarrow \mathcal{P}^0(\mathbb{Z}_p) \longrightarrow \mathbb{Q}_p \otimes \mathcal{P}^0(\mathbb{Z}_p)$$

We call the fibres of this map the *higher torsion packets* on X.

<u>Theorem "Manin–Mumford for torus-torsors" (B., in progress):</u>
If X is a hyperbolic curve, then all higher torsion packets in X are finite.

A case of non-abelian Stoll's Conjecture

Theorem (B., in progress):

Non-abelian Stoll's Conjecture holds for $X(\mathbb{Q}_p)_{\mathrm{QC}}$ when $r = 0$.

Most interesting when X is a once-punctured elliptic curve.

Unexpected points on once-punctured elliptic curves

joint with Jennifer Balakrishnan

Once-punctured elliptic curves

$X = E \setminus \{\infty\}$ with E an elliptic curve of rank 0.

⇒ exists a finite subscheme $Z \subset X$ with

$$X(\mathbb{Z}_p)_2 = X(\mathbb{Z}_p)_{\mathrm{QC}} = Z(\mathbb{Q}_p)$$

for all primes p of good reduction.

What is Z? What kinds of irrational algebraic points can it contain? We call these *unexpected points*.

Torsion orders of unexpected points

- Every unexpected point is a torsion point on E (Bianchi, '20).
- Our new description of $X(\mathbb{Z}_p)_2$ gives an explicit, purely algebraic description in terms of division polynomials, independent of p.

Theorem (Balakrishnan–B., in progress):

Suppose that Q is an unexpected point on X, of order $N \leq 50$. Then

$$N \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 12\} \cup \{15\} .$$

Example #5

(Balakrishnan–B.)

X is the once-punctured elliptic curve with equation

$$y^2 + xy = x^3 - 262888x - 62568608 .$$

rank 0

Then the 15-torsion points with x-coordinate $-123 \pm 295\sqrt{5}$ lie in $X(\mathbb{Z}_p)_2$ whenever they are $\mathbb{Q}_p$-rational.

Happy Birthday Minhyong!