Problem Set 1: Groupoids and profinite étale fundamental groups

1. In this exercise, we verify that several basic properties of groups carry over to groupoids. Let Π be a groupoid on vertex-set V.

   (a) (Uniqueness of identities) Suppose that $1_x' \in \Pi(x,x)$ is an element such that
   $$\gamma_1 \cdot 1_x' = \gamma_1 \quad \text{and} \quad 1_x' \cdot \gamma_2 = \gamma_2$$
   for all $\gamma_1 \in \Pi(x,y)$ and all $\gamma_2 \in \Pi(w,x)$ for all $y, w \in V$. Show that we have $1_x' = 1_x$.

   (b) (Uniqueness of inverses) Suppose that $\gamma \in \Pi(x,y)$ and $\gamma' \in \Pi(y,x)$ are elements such that
   $$\gamma' \cdot \gamma = 1_x \quad \text{and} \quad \gamma \cdot \gamma' = 1_y \,.$$
   Show that we have $\gamma' = \gamma^{-1}$.

2. Let Π be a groupoid on vertex-set V.

   (a) Show that if $\Pi(x,y) \neq \emptyset$ then there is a *canonical* bijection
   $$\{\text{normal subgroups } N_x \trianglelefteq \Pi(x)\} \cong \{\text{normal subgroups } N_y \trianglelefteq \Pi(y)\} \,.$$
   (You should describe this bijection explicitly.) Show that under this correspondence, the normal subgroups of finite index in $\Pi(x)$ correspond to the normal subgroups of finite index in $\Pi(y)$.

   (b) Show that if normal subgroups $N_x \trianglelefteq \Pi(x)$ and $N_y \trianglelefteq \Pi(y)$ correspond under the above bijection, then there is a canonical bijection
   $$\Pi(x,y)/N_x \cong N_y \backslash \Pi(x,y) \,.$$

   (c) Recall that the profinite completion $\Pi^\wedge$ of Π was defined by
   $$\Pi(x,y)^\wedge = \varprojlim_{N_x} (\Pi(x,y)/N_x) = \varprojlim_{N_y} (N_y \backslash \Pi(x,y)) \,,$$
   where the first inverse limit is taken over all finite index normal subgroups $N_x \trianglelefteq \Pi(x)$ and the second is taken over all finite index normal subgroups $N_y \trianglelefteq \Pi(y)$. Show carefully that the composition law

on Π induces a composition law on $\Pi^\wedge$ making it into a groupoid on object-set V. Show moreover that the composition law and inversion maps in $\Pi^\wedge$ are continuous for the natural profinite topology on the sets $\Pi(x,y)^\wedge$.

3. Let $X = \mathbb{G}_m$ be the multiplicative group over an algebraically closed field $\overline{K}$ of characteristic 0. A theorem of Serre says that the connected finite étale coverings of X, up to isomorphism, are given by the nth power maps

$$[n]\colon \mathbb{G}_m \to \mathbb{G}_m\,.$$

Based on this, what is the étale fundamental group of X based at 1? And for $x \in \overline{K}^\times$, what is the torsor of paths $\pi_1^{\text{ét}}(\mathbb{G}_m;1,x)$? Try to give your answer in as canonical a form as you can.

4. Profinite étale fundamental groups in positive characteristic are in general more complicated than in characteristic 0. Let k be a field, and consider the affine line $\mathbb{A}^1_k = \mathrm{Spec}(k[t])$ over k.

   (a) If k is algebraically closed of characteristic 0, show that $\pi_1^{\text{ét}}(\mathbb{A}^1_k;0) = 1$ is the trivial group (by combining results from class).

   (b) If however k is algebraically closed of positive characteristic p, then there are finite étale coverings of $\mathbb{A}^1_k$ given by the *Artin–Schreier coverings*

   $$X_f := \mathrm{Spec}(k[t,x]/(x^p - x - f(t)))$$

   for some element $f(t) \in k[t]$. (You may take it as given that the projection $X_{f(t)} \to \mathbb{A}^1_k = \mathrm{Spec}(k[t])$ is finite étale.)

   - Show that X_{f_1} and X_{f_2} are isomorphic as coverings of $\mathbb{A}^1_k$ if and only if $a_1 f_1(t) - a_2 f_2(t) = g(t)^p - g(t)$ for some $g(t) \in k[t]$ and $a_1, a_2 \in \mathbb{F}_p^\times$.
   - Show that X_f is connected if and only if $f(t)$ is not of the form $g(t)^p - g(t)$ for $g(t) \in k[t]$.
   - Deduce that $\pi_1^{\text{ét}}(\mathbb{A}^1_k) \neq 1$ is not the trivial group.
   - Show that if k'/k is a non-trivial extension of algebraically closed fields of characteristic p, then the induced map

     $$\pi_1^{\text{ét}}(\mathbb{A}^1_{k'};0) \to \pi_1^{\text{ét}}(\mathbb{A}^1_k;0)$$

     is not an isomorphism.