Math 283Z Final Exam: Solutions

Problem 1

Let U be a pro-unipotent group over a field F of characteristic 0.

1. Write down the universal property satisfied by the abelianisation U^{ab} of U.
2. Show that if $S \subseteq U(F)$ is a set of F-rational points on U, then the subgroup generated by S is Zariski-dense in U if and only if the subgroup generated by its image in U^{ab} is Zariski-dense. Deduce that U is finitely generated (meaning that there exists a *finite* set S such that the subgroup it generates is Zariski-dense) if and only if U^{ab} is finite-dimensional, i.e. a vector group.
3. Show that if $\phi\colon U_1 \to U_2$ is a homomorphism of pro-unipotent groups such that $\phi^{\mathrm{ab}}\colon U_1^{\mathrm{ab}} \to U_2^{\mathrm{ab}}$ is surjective, then ϕ is surjective. Show moreover that if ϕ^{ab} is an isomorphism and U_2 is free, then ϕ is an isomorphism.

Results from class may be used without proof, provided that they are clearly and precisely stated.

Solution

1. The abelianisation U^{ab} is an abelian pro-unipotent group with a homomorphism $\phi\colon U \to U^{\mathrm{ab}}$ of pro-unipotent groups. This satisfies the following universal property: for any homomorphism $\psi\colon U \to U'$ of pro-unipotent groups with U' abelian, there exists a unique homomorphism $\psi'\colon U^{\mathrm{ab}} \to U'$ of pro-unipotent groups making the following triangle commute

$$\begin{tikzcd} U \arrow[r, "\phi"] \arrow[dr, "\psi"'] & U^{\mathrm{ab}} \arrow[d, "\psi'"] \\ & U' . \end{tikzcd}$$

2. Let $U_0 \subseteq U$ be the Zariski-closure of the subgroup generated by S. U_0 is a subgroup-scheme of U, and so is also pro-unipotent. The Zariski-closure of the subgroup generated by the image of S in U^{ab} is equal to the scheme-theoretic image of $U_0 \to U^{\mathrm{ab}}$. So we want to prove that $U_0 = U$ if and only if $U_0 \to U^{\mathrm{ab}}$ is surjective as a homomorphism of group-schemes.

Using the equivalence between pro-unipotent groups and pro-nilpotent Lie algebras and using an inverse limit argument, it suffices to prove that if $\mathfrak{u}$ is a finite-dimensional nilpotent Lie algebra and $\mathfrak{u}_0 \leq \mathfrak{u}$ is a Lie subalgebra, then $\mathfrak{u}_0 = \mathfrak{u}$ if and only if $\mathfrak{u}_0$ surjects onto $\mathfrak{u}^{\mathrm{ab}}$. The "only if" direction is immediate; we focus on the "if" direction. We proceed by induction over the length of the descending central series on $\mathfrak{u}$. Suppose that $\Gamma^{n+1}\mathfrak{u} = 0$ for some $n \geq 2$ and we know the result for all Lie subalgebras of $\mathfrak{u}/\Gamma^n\mathfrak{u}$. Since $\mathfrak{u}_0$ surjects onto $\mathfrak{u}^{\mathrm{ab}} = (\mathfrak{u}/\Gamma^n\mathfrak{u})^{\mathrm{ab}}$, we know by the inductive hypothesis that it surjects onto $\mathfrak{u}/\Gamma^n\mathfrak{u}$. But we know that the Lie bracket

$$\mathfrak{u}^{\mathrm{ab}} \otimes_F \frac{\Gamma^{n-1}\mathfrak{u}}{\Gamma^n\mathfrak{u}} \to \Gamma^n\mathfrak{u}$$

is surjective. Since $\mathfrak{u}_0$ surjects onto $\mathfrak{u}^{\mathrm{ab}}$ and $\mathfrak{u}/\Gamma^n\mathfrak{u}$, it follows by taking Lie brackets of appropriate elements of $\mathfrak{u}_0$ that it contains $\Gamma^n\mathfrak{u}$. So $\mathfrak{u}_0 = \mathfrak{u}$ and the induction is complete.

For the part regarding finite generation, if U is finitely generated then so too is the pro-vector group U^{ab}, which is thus a vector group. Conversely, if U^{ab} is a vector group, then a vector space basis of $U^{\mathrm{ab}}(F)$ is a finite subset which generates a Zariski-dense subgroup. So choosing any lift of this basis to $U(F)$ gives a finite generating set for U.

3. Suppose that ϕ^{ab} is surjective. If we let U_0 denote the scheme-theoretic image of ϕ, then U_0 surjects onto U_2^{ab} and so $U_0 = U_2$ by the first part. So ϕ is surjective.

   Now suppose that ϕ^{ab} is an isomorphism and U_2 is free. So ϕ is surjective by the first part, and freeness of U_2 implies that ϕ has a left inverse $\psi\colon U_2 \to U_1$ such that $\phi \circ \psi = 1_{U_2}$. Now ψ^{ab} is the inverse of ψ^{ab}, so is surjective and ψ is surjective by the first part again. Since ψ is surjective and has a right inverse, it is an isomorphism and so ϕ was already an isomorphism.

Problem 2

Let F be a characteristic 0 field. Let $\mathcal{T}$ denote the following $\otimes$-category. The objects of $\mathcal{T}$ are finite-dimensional graded vector spaces

$$V = \bigoplus_{i \in \mathbb{Z}} V_i$$

over F, endowed with an endomorphism

$$N = N_V : V \to V$$

which is graded of degree -2 ($N(V_i) \subseteq N(V_{i-2})$) with the property that

$$N^i : V_i \to V_{-i}$$

is an isomorphism for all $i \geq 0$. Morphisms are defined in the obvious way. The tensor product on $\mathcal{T}$ is given by the usual tensor product $V^1 \otimes V^2$ of graded vector spaces, endowed with the endomorphism $N_{V^1 \otimes V^2} = N_{V^1} \otimes 1_{V^2} + 1_{V^1} \otimes N_{V^2}$ where 1_V denotes the identity on V. The tensor unit is $\mathbf{1} = F$, placed in degree 0 for the grading, with $N_{\mathbf{1}} = 0$. You may use without proof the fact that this $\otimes$-category is neutral Tannakian over F, and that the forgetful functor is a fibre functor.

1. Show that for all $n \geq 1$ there is a unique simple object of $\mathcal{T}$ of dimension n (simple means that it has no subobjects in $\mathcal{T}$ other than itself and the zero subobject). Describe this object explicitly. Show moreover that every object of $\mathcal{T}$ decomposes as a direct sum of simple objects.

2. Let V be a representation of SL_2.

   (a) Let $\mathbb{T} \subset \mathrm{SL}_2$ denote the algebraic subgroup consisting of elements of the form
   $$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$
   so that $\mathbb{T}$ is isomorphic to the multiplicative group $\mathbb{G}_m$. Briefly describe (without proof) how the action of $\mathbb{T}$ on V corresponds to a grading on V.

   (b) Let $\mathbb{U} \subset \mathrm{SL}_2$ denote the algebraic subgroup consisting of elements of the form
   $$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix},$$
   so that $\mathbb{U}$ is isomorphic to the additive group $\mathbb{G}_a$. Briefly describe (without proof) how the action of $\mathbb{U}$ on V corresponds to a nilpotent endomorphism N of V.

   (c) Show that the above grading and nilpotent endomorphism make V into an object of $\mathcal{T}$.

3. Prove that the above construction defines a $\otimes$-equivalence from the category of SL_2-representations to $\mathcal{T}$. Conclude that the Tannaka group of $\mathcal{T}$, based at the forgetful functor, is isomorphic to SL_2.

Standard results about the representation theory of SL_2 may be used without proof, provided that they are clearly and precisely stated. You may look up references for the representation theory of SL_2; if you do, you must include inline citations where these results are used.

Solution

1. For $n \geq 0$ let $V^{(n)}$ denote the $(n+1)$-dimensional vector space with basis

$$e_n, e_{n-2}, e_{n-4}, \ldots, e_{-n}\,.$$

We endow $V^{(n)}$ with the grading placing e_i in degree i, and define an endomorphism N of $V^{(n)}$ by

$$N(e_i) := \begin{cases} e_{i-2} & i \neq -n\,, \\ 0 & i = -n\,. \end{cases}$$

This makes $V^{(n)}$ into an object of $\mathcal{T}$. We claim that it is simple. If $V \leq V^{(n)}$ were a non-zero subobject, then since it is a graded subspace, it must contain some e_i. Applying the operator N some number of times if necessary, we deduce that $e_{-n} \in V_{-n}$. So we also have $V_n \neq 0$, and so $e_n \in V_n$. Applying N again, we see that $e_i \in V_i$ for all i, and so $V = V^{(n)}$. This completes the proof that $V^{(n)}$ is simple.

Next we show that every object of $\mathcal{T}$ is a direct sum of copies of the objects $V^{(n)}$. This proves that every object is completely reducible, and that the $V^{(n)}$ are the only simple objects. For this, suppose that V is a non-zero object of $\mathcal{T}$, and let n be the largest non-negative integer such that $V_n \neq 0$. Choose a non-zero element $e_n \in V_n$. Since V is an object of $\mathcal{T}$, we know that $N^n(e_n) \neq 0$, and so the elements $N^j(e_n)$ are non-zero elements of V_{n-2j} for $0 \leq j \leq n$. Moreover, maximality of n ensures that $V_{-n-2} = 0$, and so $N^{n+1}(e_n) = 0$. So the subspace spanned by $N^j(e_n)$ for $0 \leq j \leq n$ is a subobject of V isomorphic to $V^{(n)}$.

We now want to show that this copy of $V^{(n)}$ has a complement. Choose a vector space complement W_{-n} to the space spanned by $N^n(e_n)$ inside V_{-n}. Then define subspaces $W_i \leq V_i$ by

$$W_{2j-n} = (N^j)^{-1}(W_{-n}) = \{e \in V_{2j-n} \,:\, N^j(e) \in W_{-n}\}$$

for $0 \leq j \leq n$, and $W_i = V_i$ otherwise. It is easy to see that W_{2j-n} is a vector space complement to the space spanned by $N^{n-j}(e_n)$ inside V_{2j-n} for all $0 \leq j \leq n$, and so $V = V^{(n)} \oplus W$ as vector spaces, where $W = \bigoplus_i W_i$. Moreover, W is clearly stable under N and has the property that $N^i\colon W_i \to W_{-i}$ is an isomorphism for all $i \geq 0$ (e.g. it is certainly injective, and the dimensions match). So W is a subobject in $\mathcal{T}$ and we have the desired complement.

2. (a) The action of $\mathbb{T}$ corresponds to the grading $V = \bigoplus_i V_i$, where the matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ acts on V_i by multiplication by λ^i.

   (b) Since $\mathbb{U}$ is unipotent, it acts on V unipotently. Taking the logarithm of this action shows that there is a unique nilpotent endomorphism N of V such that
$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \cdot v = e^{\alpha N}(v)$$
   for all $v \in V$.

   (c) If V^1 and V^2 are two representations of SL_2, then it is easy to check that the grading on $V^1 \otimes_F V^2$ coming from the action of $\mathbb{T}$ is the tensor product of the gradings on V^1 and V^2, and that the endomorphism $N_{V^1 \otimes V^2}$ coming from the action of $\mathbb{U}$ is equal to $N_{V^1} \otimes 1_{V^2} + 1_{V^1} \otimes N_{V^2}$. In other words, the above descriptions are defining a $\otimes$-functor G from the $\otimes$-category of representations of SL_2 to the $\otimes$-category of finite-dimensional graded vector spaces with a nilpotent endomorphism. We want to show that the image of this functor is contained in $\mathcal{T}$.

   Now every representation of SL_2 is a direct sum of copies of $\mathrm{Sym}^n(\mathrm{std})$, where std is the standard 2-dimensional representation of SL_2. Since the functor G commutes with direct sums and symmetric powers, it thus suffices to show that $G(\mathrm{std}) \in \mathcal{T}$. But this is easy: std has a basis $\{e_1, e_{-1}\}$ where the action of SL_2 is given by
$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \cdot e_1 = \lambda e_1 \,, \qquad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \cdot e_{-1} = \lambda^{-1} e_{-1} \,,$$
$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \cdot e_1 = e_1 + \alpha e_{-1} \,, \qquad \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \cdot e_{-1} = e_{-1} \,.$$
   So $e_{\pm 1}$ is in degree ± 1 for the grading, and the endomorphism N acts via $N(e_1) = e{-1}$ and $N(e_{-1}) = 0$. This makes $G(\mathrm{std})$ into an object of $\mathcal{T}$ (namely, it is isomorphic to $V^{(1)}$).

3. To show that the functor G above is a $\otimes$-equivalence, we first note that
$$V^{(n)} \cong \mathrm{Sym}^n(V^{(1)})$$
for all $n \geq 0$. Indeed, $\mathrm{Sym}^n(V^{(1)})$ has a basis given by the elements $\frac{n!}{(n-j)!} e_1^{n-j} e_{-1}^j$ for $0 \leq j \leq n$, which are acted on by N by
$$N\left(\frac{n!}{(n-j)!} e_1^{n-j} e_{-1}^j\right) = \frac{n!}{(n-j-1)!} e_1^{n-j-1} e_{-1}^{j+1}$$

for $0 \le j < n$. So $\mathrm{Sym}^n(V^{(1)})$ is isomorphic to $V^{(n)}$ (via the isomorphism taking e_{n-2j} to $\frac{n!}{(n-j)!}e_1^{n-j}e_{-1}^j$).

Thus, we know that

$$G(\mathrm{Sym}^n(\mathrm{std})) \cong \mathrm{Sym}^n(G(\mathrm{std})) \cong \mathrm{Sym}^n(V^{(1)}) \cong V^{(n)}$$

for all $n \ge 0$. In light of the complete reducibility of objects of $\mathcal{T}$, this means that G is essentially surjective. It remains to show that it is fully faithful, i.e. that the map

$$\mathrm{Hom}_{\mathrm{SL}_2}(V^1, V^2) \to \mathrm{Hom}_{\mathcal{T}}(G(V^1), G(V^2)) \tag{$*$}$$

is bijective for all representations V^1 and V^2 of SL_2. Since all representations of SL_2 are completely irreducible, one immediately reduces to the case that V^1 and V^2 are irreducible representations of SL_2, i.e. that $V^1 = \mathrm{Sym}^{n_1}(\mathrm{std})$ and $V^2 = \mathrm{Sym}^{n_2}(\mathrm{std})$ for some $n_1, n_2 \ge 0$. If $n_1 \ne n_2$, then both sides of $(*)$ are zero (no homomorphisms between nonisomorphic simple objects), while if $n_1 = n_2$ then both sides are one-dimensional, spanned by the class of the identity by Schur's Lemma. This concludes the proof that G is fully faithful, hence a $\otimes$-equivalence.

Problem 3

Let $E/\mathbb{Q}$ be an elliptic curve, and let $Y = E \setminus \{0\}$ be the complement of the identity in E. Let U be the $\mathbb{Q}_p$-pro-unipotent étale fundamental group of Y based at some rational basepoint b. Let

$$U = \mathrm{W}_{-1}U \trianglerighteq \mathrm{W}_{-2}U \trianglerighteq \dots$$

denote the weight filtration on U (which is equal to the descending central series up to reindexing), and adopt the usual notation

$$U_n := \frac{U}{\mathrm{W}_{-n-1}U} \quad \text{and} \quad V_n := \frac{\mathrm{W}_{-n}U}{\mathrm{W}_{-n-1}U} \,.$$

1. What is the dimension of V_1 and V_2?
2. Show that the commutator pairing in the group U_2 induces an isomorphism

   $$\bigwedge^2 V_1 \cong V_2$$

   of $G_{\mathbb{Q}}$-representations. Using this, describe the Galois action on V_2. [Hint: use the Weil pairing of the elliptic curve E, recalling that V_1 is the $\mathbb{Q}_p$-linear Tate module of E.]
3. Now assume that E has Mordell–Weil rank 1 and finite Tate–Shafarevich group. Determine the dimensions of the global Bloch–Kato Selmer groups

   $$\mathrm{H}^1_f(G_{\mathbb{Q}}, V_1) \quad \text{and} \quad \mathrm{H}^1_f(G_{\mathbb{Q}}, V_2)\,.$$

   Determine also the dimensions of the local Bloch–Kato Selmer groups

   $$\mathrm{H}^1_f(G_p, V_1) \quad \text{and} \quad \mathrm{H}^1_f(G_p, V_2)\,.$$

4. Under the above assumptions, conclude that $\mathcal{Y}(\mathbb{Z})$ is finite for every integral model $\mathcal{Y}$ of Y. [Use the Chabauty–Kim criterion for the quotient U_2.]

Results from class may be used without proof, provided that they are clearly and precisely stated.

Solution

1. By comparison with the complex numbers, we know that U is the free $\mathbb{Q}_p$-pro-unipotent group on two generators (since $Y(\mathbb{C})$ is a 2-dimensional torus minus a single point). Moreover, since $\mathrm{W}_{-2}U$ is the kernel of the map

   $$U \to \pi_1^{\mathbb{Q}_p}(E_{\overline{\mathbb{Q}}}; b)^{\mathrm{ab}} = \pi_1^{\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}; b)^{\mathrm{ab}}\,,$$

   it follows that $\mathrm{W}_{-2}U = \Gamma^2 U$, and thence that $\mathrm{W}_{-k}U = \Gamma^k U$ for all $k \geq 1$. So the weight filtration on U is, up to reordering, the same as the descending central series. We already computed in class the dimensions of the graded pieces of the descending central series on the free pro-unipotent group on two generators: this gives

$$\dim(V_1) = 2 \quad \text{and} \quad \dim(V_2) = 1\,.$$

2. We know that the commutator pairing is a $G_{\mathbb{Q}}$-equivariant surjection; since both sides have the same dimension, it is an isomorphism. Now we have

$$V_2 \cong \bigwedge^2 V_1 \cong \bigwedge^2 V_pE \cong \mathbb{Q}_p(1)$$

where V_pE is the $\mathbb{Q}_p$-linear Tate module of E, using the Weil pairing for the final identification. So the action on V_2 is the same as on $\mathbb{Q}_p(1)$, i.e. via the cyclotomic character.

3. Results we saw in class at various points give

$$\begin{aligned} \dim \mathrm{H}^1_f(G_{\mathbb{Q}}, V_1) &= r = 1 \quad & \dim \mathrm{H}^1_f(G_{\mathbb{Q}}, V_2) &= \dim \mathrm{H}^1_f(G_{\mathbb{Q}}, \mathbb{Q}_p(1)) = 0 \\ \dim \mathrm{H}^1_f(G_p, V_1) &= g = 1 \quad & \dim \mathrm{H}^1_f(G_p, V_2) &= \dim \mathrm{H}^1_f(G_p, \mathbb{Q}_p(1)) = 1\,. \end{aligned}$$

(Finiteness of the Tate–Shafarevich group is used in the first equality above.)

4. The above dimension calculations show that

$$\dim \mathrm{H}^1_f(G_{\mathbb{Q}}, V_1) + \dim \mathrm{H}^1_f(G_{\mathbb{Q}}, V_2) < \dim \mathrm{H}^1_f(G_p, V_1) + \dim \mathrm{H}^1_f(G_p, V_2)\,,$$

and so the Chabauty–Kim criterion for the quotient U_2 gives finiteness of $\mathcal{Y}(\mathbb{Z})$.

Problem 4

Write an essay on the profinite étale fundamental group, explaining how Grothendieck generalised the notion of fundamental groups of topological spaces to schemes. Your essay should include:

- A definition of the profinite étale fundamental group, including the structure maps.
- A definition of the topology on the étale fundamental group, and a proof that it is profinite. (You do not need to prove that the group operations are continuous.)
- A statement and proof of the comparison theorem with the usual fundamental group of schemes of finite type over $\mathbb{C}$.
- One or more examples where you determine the fundamental group/oid of a scheme.

You may assume that your reader is familiar with the basic language of algebraic geometry, including the definition and basic properties of finite and étale morphisms. Results which were not proved in class (e.g. the Riemann Existence Theorem for coverings) do not need to be proved, but you should clearly and precisely state any results you rely on.

Solution

[Omitted.]