Problem Set 6: The Chabauty–Kim criterion

1. Let $Y = \mathbb{P}^1_{\mathbb{Q}} \setminus \{\pm i, \infty\} = \mathrm{Spec}(\mathbb{Q}[t, \frac{1}{t^2+1}])$ denote the complement of the divisor consisting of the points i, $-i$ and ∞ inside $\mathbb{P}^1_{\mathbb{Q}}$. (Note that this divisor is defined over $\mathbb{Q}$, even though the points i and $-i$ are not individually defined over $\mathbb{Q}$.) Let $\mathcal{Y} = \mathbb{P}^1_{\mathbb{Z}} \setminus \{\pm i, \infty\} = \mathrm{Spec}(\mathbb{Z}[t, \frac{1}{t^2+1}])$ be its standard integral model over $\mathbb{Z}$, and let S be a finite set of prime numbers. Let U denote the $\mathbb{Q}_p$-pro-unipotent étale fundamental group of $Y_{\overline{\mathbb{Q}}}$, and V_{2n} its graded pieces with respect to the weight filtration, as usual.

   (a) Explain briefly why U is profinite free on two generators and $V_n = 0$ for n odd.

   (b) Show that $V_2 = \mathbb{Q}_p(1) \oplus \mathbb{Q}_p(\chi)(1)$, where $\chi\colon G_{\mathbb{Q}} \to \{\pm 1\}$ is the quadratic character associated to the extension $\mathbb{Q}(i)/\mathbb{Q}$. (Here, $\mathbb{Q}_p(\chi)$ denotes the one-dimensional vector space on which $G_{\mathbb{Q}}$ acts via the character χ, and $V(1)$ is shorthand for $V \otimes \mathbb{Q}_p(1)$.)

   (c) Show that for $n \geq 1$ there are integers r_n^+ and r_n^- such that

   $$V_{2n} = \mathbb{Q}_p(n)^{\oplus r_n^+} \oplus \mathbb{Q}_p(\chi)(n)^{\oplus r_n^-}\,.$$

   (d) Show that $r_n^- \geq 1$ for all $n \geq 1$.

   (e) If p is a prime split in $\mathbb{Q}(i)$, explain briefly why

   $$\dim \mathrm{H}^1_f(G_p, \mathbb{Q}_p(\chi)(n)) = 1$$

   for $n \geq 1$. (Optional: prove the same for any p.)

   (f) A result of Soulé[1] says that for any number field K and odd p, we have

   $$\dim \mathrm{H}^1(G_K, \mathbb{Q}_p(n)) = \begin{cases} r_K + s_K & \text{if } n \geq 3 \text{ odd,} \\ s_K & \text{if } n \geq 2 \text{ even,} \end{cases}$$

   where r_K denotes the number of real embeddings of K and s_K denotes the number of pairs of conjugate complex embeddings of K. Using this result, show that

   $$\dim \mathrm{H}^1(G_{\mathbb{Q}}, \mathbb{Q}_p(\chi)(n)) = \begin{cases} 0 & \text{if } n \geq 3 \text{ odd,} \\ 1 & \text{if } n \geq 2 \text{ even,} \end{cases}$$

   (g) Applying the Chabauty–Kim criterion to the quotient U_{2N} for $N \gg 0$, show that $\mathcal{Y}(\mathbb{Z}_S)$ is finite. Conclude that for any finite set S, there are only finitely many integers a such that all prime factors of a^2+1 lie in S.

[1] This is the same result we used in class, just stated more generally.