# Problem Set 5: Cohomology schemes and the Bloch–Kato exponential

[Warning: one of the questions here is stated a little inaccurately, and is missing a condition. It is your task to identify this missing condition and restore it.]

1. Let $U$ be a unipotent group over $\mathbb{Q}_p$ and let $G$ be a finite group acting on $U$, necessarily continuously. Show that $\mathrm{H}^1(G, U(\Lambda)) = \{*\}$ for every $\mathbb{Q}_p$-algebra $\Lambda$. (This explains why the cohomology of unipotent groups with an action of $G_\mathbb{R}$ or $G_\mathbb{C}$ is not so interesting.)

2. Let $U$ be a finitely generated pro-unipotent group over $\mathbb{Q}_p$ endowed with a continuous action of a profinite group $G$ and a $G$-stable separated filtration $\mathrm{W}_\bullet U$. Set $U_n \coloneqq U/\mathrm{W}_{-n-1}U$ as usual. Show that the natural map
$$\mathrm{H}^1(G, U) \to \varprojlim_n \mathrm{H}^1(G, U_n)$$
is an isomorphism of functors. (This is in the notes, but we skipped the proof in class.)

3. Let $U$ be a unipotent group over a characteristic 0 field $F$ and let $U^+ \leq U$ be a subgroup-scheme (so $U^+$ is also unipotent). Let $V \leq \mathrm{Lie}(U)$ be a complement of $\mathrm{Lie}(U^+)$, i.e. $\mathrm{Lie}(U) = \mathrm{Lie}(U^+) \oplus V$.

   (a) Show that the map
   $$U^+ \times \mathbb{A}(V) \to U$$
   given by
   $$(u^+, v) \mapsto u^+ \cdot \exp(v)$$
   is an isomorphism of varieties over $F$. Deduce that the functor
   $$U^+\backslash U \colon \mathbf{Alg}_F \to \mathbf{Set}$$
   sending an $F$-algebra $\Lambda$ to $U^+(\Lambda)\backslash U(\Lambda)$ is representable by an affine space of dimension $\dim(U) - \dim(U^+)$.

   (b) The previous part provides an isomorphism
   $$U^+\backslash U \cong \mathbb{A}(\mathrm{Lie}(U^+)\backslash \mathrm{Lie}(U))$$
   of functors. Show that in general this isomorphism is non-canonical, i.e. it depends on the complement $V$ chosen above. (So if we want say that $U^+\backslash U$ is isomorphic to the specific affine space $\mathbb{A}(\mathrm{Lie}(U^+)\backslash \mathrm{Lie}(U))$, then we should be careful, because we don't have a canonical choice of isomorphism.)

4. Let $K$ be a finite extension of $\mathbb{Q}_p$.

(a) Show that
$$\dim_{\mathbb{Q}_p} \mathrm{H}^1(G_K, \mathbb{Q}_p(1)) = [K : \mathbb{Q}_p] + 1 \,.$$

(b) Using the Bloch–Kato exponential exact sequence, find $\dim_{\mathbb{Q}_p} \mathrm{H}^1_e(G_K, \mathbb{Q}_p(1))$.

(c) Find $\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(G_K, \mathbb{Q}_p(1))$ and $\dim_{\mathbb{Q}_p} \mathrm{H}^1_g(G_K, \mathbb{Q}_p(1))$.