Problem Set 4: Galois action and non-abelian cohomology

1. Let $Y = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ be the thrice-punctured line over $\mathbb{Q}$ and let $x_0 \in Y(\mathbb{Q})$ be any rational basepoint. Let $U = \pi_1^{\mathbb{Q}_p}(Y_{\overline{\mathbb{Q}}}; x_0)$ be the $\mathbb{Q}_p$-pro-unipotent étale fundamental group of $Y_{\overline{\mathbb{Q}}}$. By considering the map $Y \to \mathbb{G}_m^2$ given by $z \mapsto (z, 1-z)$, or otherwise, show that U^{ab} is $G_{\mathbb{Q}}$-equivariantly isomorphic to the vector group associated to $\mathbb{Q}_p(1)^2$. Using this, determine the graded pieces of the weight filtration on U (you don't need to give a closed form expression for their dimensions, but you should be specific about their Galois actions).

2. Let X be a hyperelliptic curve over a characteristic 0 field K, and let $x_0 \in X(K)$ be a K-rational Weierstrass point. Let $U = \pi_1^{\mathbb{Q}_p}(X_{\overline{K}}; x_0)$ be the $\mathbb{Q}_p$-pro-unipotent étale fundamental group, let $U_n \coloneqq U/\Gamma^{n+1}U$ denote the nth quotient by the descending central series, and let $V_n = \Gamma^n U/\Gamma^{n+1}U$ denote the nth graded piece of the descending central series, so that the sequence
$$1 \to V_n \to U_n \to U_{n-1} \to 1$$
is a G_K-equivariant central extension. Show that for $n = 2$, the sequence
$$0 \to V_2 \to \mathrm{Lie}(U_2) \to \mathrm{Lie}(U_1) \to 0$$
on Lie algebras splits as a sequence of G_K-representations. [Hint: use the hyperelliptic involution on X.]

3. Let Π and G be topological groups, and endow Π with the *trivial* G-action (every element of G acts as the identity). Show that
$$\mathrm{H}^1(G, \Pi) = \mathrm{Hom}^{\mathrm{out}}(G, \Pi) = \mathrm{Hom}(G, \Pi)/\Pi$$
is the set of continuous *outer homomorphisms* $G \to \Pi$, i.e. the set of continuous group homomorphisms modulo the conjugation action of Π.

   Now suppose that $G = \mathbb{Z}^2$ and $\Pi = D_8$ is the dihedral group of order eight. Show that there is no way to put a group structure on $\mathrm{H}^1(G, \Pi)$ and $\mathrm{H}^1(G, \Pi^{\mathrm{ab}})$ for which the map
$$\mathrm{H}^1(G, \Pi) \to \mathrm{H}^1(G, \Pi^{\mathrm{ab}})$$
is a group homomorphism. (This shows that there is really no hope for putting a sensible group structure on non-abelian cohomology.)

4. Let

$$1 \to Z \to \Pi \to Q \to 1$$

be a G-equivariant topologically split short exact sequence of topological groups endowed with continuous actions of G. Show that there is a coboundary map

$$\delta^0 \colon \mathrm{H}^0(G,Q) \to \mathrm{H}^1(G,Z)$$

(a map of pointed sets) for which the sequence

$$1 \to \mathrm{H}^0(G,Z) \to \mathrm{H}^0(G,\Pi) \to \mathrm{H}^0(G,Q) \xrightarrow{\delta^0} \mathrm{H}^1(G,Z) \to \mathrm{H}^1(G,\Pi) \to \mathrm{H}^1(G,Q)$$

is exact. Show moreover that there is a right action of $\mathrm{H}^0(G,Q)$ on $\mathrm{H}^1(G,Z)$ whose orbits are exactly the fibres of $\mathrm{H}^1(G,Z) \to \mathrm{H}^1(G,\Pi)$ and such that the stabiliser of the distinguished point of $\mathrm{H}^1(G,Z)$ is the image of $\mathrm{H}^0(G,\Pi) \to \mathrm{H}^0(G,Q)$.

5. Let Π be a connected groupoid in topological spaces for which each $\Pi(x,y)$ is endowed with a continuous action of a topological group G in a manner compatible with composition, identities and inversion.

   - Let x_0, y_0 be vertices of Π. Show that $\Pi(y_0)$ is G-equivariantly isomorphic to a Serre twist ${}_\xi\Pi(x_0)$ of $\Pi(x_0)$ for some $\xi \in \mathrm{Z}^1(G,\Pi(x_0))$. (This isomorphism will depend on some choices, which you should specify.)
   - Show that the composite isomorphism

     $$\mathrm{H}^1(G,\Pi(y_0)) \cong \mathrm{H}^1(G,{}_\xi\Pi(x_0)) \cong \mathrm{H}^1(G,\Pi(x_0))$$

     is independent of any choices. If ϕ_{x_0,y_0} denotes this composite isomorphism, show moreover that it satisfies the cocycle condition

     $$\phi_{x_0,z_0} = \phi_{x_0,y_0} \circ \phi_{y_0,z_0}$$

     for all vertices x_0, y_0, z_0. (So the identifications $\mathrm{H}^1(G,\Pi(y_0)) \cong \mathrm{H}^1(G,\Pi(x_0))$ are canonical and coherent.)
   - Show that the abstract non-abelian Kummer maps $j \colon V(\Pi) \to \mathrm{H}^1(G,\Pi(x_0))$ associated to Π are independent of the choice of basepoint x_0, in the sense that the square

     $$\begin{array}{ccc} V(\Pi) & = & V(\Pi) \\ \downarrow j & & \downarrow j \\ \mathrm{H}^1(G,\Pi(y_0)) & \xrightarrow[\sim]{\phi_{x_0,y_0}} & \mathrm{H}^1(G,\Pi(x_0)) \end{array}$$

     commutes for all vertices x_0, y_0.