Problem Set 3: Malčev completion and the Tannakian formalism

1. Let Π be a finitely generated *abelian* profinite group. Show that the $\mathbb{Q}_p$-Malčev completion of Π is the vector group associated to $\mathbb{Q}_p \otimes_{\hat{\mathbb{Z}}} \Pi$.

2. Let Π be a finitely generated profinite group, and let Π' be the largest torsion-free pro-p quotient of Π. Show that the natural map $\Pi_{\mathbb{Q}_p} \to \Pi'_{\mathbb{Q}_p}$ on $\mathbb{Q}_p$-Malčev completions is an isomorphism.

3. Let Π be the profinite group

$$\Pi = \mathbb{Z}_2 \rtimes C_2 \,,$$

where in the semidirect product, the non-identity element of C_2 acts on $\mathbb{Z}_2$ by $x \mapsto -x$.

   (a) What is the abelianisation Π^{ab}?

   (b) Using this, show that $\Pi_{\mathbb{Q}_2} = 1$.

   (This shows that the Malčev completion functor is not exact, as the injective map $\mathbb{Z}_2 \hookrightarrow \Pi$ does not remain injective after Malčev completing.)

4. Which of the following $\mathbb{Q}$-linear abelian tensor categories are pre-Tannakian? And which are neutral Tannakian?

   (a) Finite dimensional vector spaces V (over $\mathbb{Q}$) with an automorphism σ_V (the tensor unit is $\mathbf{1} = \mathbb{Q}$ with $\sigma_{\mathbf{1}} = 1$, and the tensor product is the $\mathbb{Q}$-linear tensor product with $\sigma_{V_1 \otimes V_2} = \sigma_{V_1} \otimes \sigma_{V_2}$).

   (b) Finite dimensional vector spaces V with an *endo*morphism σ (same tensor structure as above).

   (c) Finite dimensional vector spaces V with an endomorphism σ (the tensor unit is $\mathbf{1} = \mathbb{Q}$ with $\sigma_{\mathbf{1}} = 0$, and the tensor product is the $\mathbb{Q}$-linear tensor product with $\sigma_{V_1 \otimes V_2} = \sigma_{V_1} \otimes 1_{V_2} + 1_{V_1} \otimes \sigma_{V_2}$).

   (d) Finite dimensional graded vector spaces (with the usual tensor structure).

   (e) Finite dimensional vector spaces V endowed with a finite decreasing filtration $\mathrm{F}^\bullet$ (with the usual tensor structure).

   (f) Local systems on a *disconnected* topological space X.

   (g) Vector spaces, possibly of infinite dimension.

5. In this exercise, we compute a simple example of a Tannakian fundamental group. Let $\mathcal{T}$ denote the category of finite-dimensional F-vector spaces V endowed with a *nilpotent* endomorphism $N = N_V \in \mathrm{End}(V)$. The morphisms in $\mathcal{T}$ are the F-linear maps which are compatible with the chosen endomorphisms. The category $\mathcal{T}$ is an F-linear neutral Tannakian category, where the tensor product $V_1 \otimes V_2$ is the usual F-linear tensor product with endomorphism $N_{V_1\otimes V_2} = N_{V_1} \otimes 1_{V_2} + 1_{V_1} \otimes N_{N_v}$. The tensor unit is $\mathbf{1} = F$ with endomorphism $N_{\mathbf{1}} = 0$. The forgetful functor $\omega\colon \mathcal{T} \to \mathbf{Vec}_F$ is a fibre functor.

   Let $\gamma \in \pi_1(\mathcal{T};\omega)(F)$ be a $\otimes$-natural automorphism of ω.

   (a) Let $V_1 \in \mathcal{T}$ be the 2-dimensional F-vector space with basis v_0, v_1, endowed with the endomorphism N defined by $N(v_0) = v_1$ and $N(v_1) = 0$. Show that the action of γ on V_1 is given by $\gamma(v_0) = v_0 + \chi(\gamma)v_1$ and $\gamma(v_1) = v_1$ for some $\chi(\gamma) \in F$. (Hint: V_1 is an extension

   $$0 \to \mathbf{1} \to V_1 \to \mathbf{1} \to 0\,.)$$

   (b) Show that if $V \in \mathcal{T}$ and if $v \in V$ is a vector such that $N^2(v) = 0$, then there is a unique morphism $V_1 \to V$ in $\mathcal{T}$ taking v_0 to v. Deduce that in this case $\gamma(v) = v + \chi(\gamma)v$ where $\chi(\gamma) \in F$ is the same as above.

   (c) Show more generally that if $V \in \mathcal{T}$ and if $v \in V$ is a vector such that $N^{m+1}(v) = 0$, then there is a (non-unique) morphism $V_1^{\otimes m} \to V$ in $\mathcal{T}$ taking $v_0^{\otimes m}$ to v. (You should describe this map explicitly.) Deduce that in this case

   $$\gamma(v) = \sum_{i=0}^{\infty} \frac{1}{i!}\chi(\gamma)^i N^i(v)\,.$$

   (The terms of the sum are zero for $i > m$.)

   (d) Deduce that the map $\chi\colon \pi_1(\mathcal{T};\omega)(F) \to F$ is an isomorphism of groups, where the group law on F is the additive group law. *Show more generally that χ defines an isomorphism

   $$\pi_1(\mathcal{T};\omega) \xrightarrow{\sim} \mathbb{G}_a$$

   of affine group-schemes over F.

6. Let F be a field and let $\mathcal{T} = (\mathcal{T}, \otimes, \mathbf{1})$ be an F-linear abelian $\otimes$-category. Let E be an object of $\mathcal{T}$ which admits a strong dual E^*. If $E' \in \mathcal{T}$ is a second object, we define the *internal hom* by

   $$\underline{\mathrm{Hom}}(E, E') := E' \otimes E^* \in \mathcal{T}\,.$$

   Show that for any third object $E'' \in \mathcal{T}$ there is a canonical bijective correspondence between morphisms

   $$E'' \otimes E \to E' \quad \text{and} \quad E'' \to \underline{\mathrm{Hom}}(E, E')\,.$$