Problem Set 2: Pro-categories, descending central series and unipotent groups

1. Consider a sequence of objects
$$\cdots \to X_3 \to X_2 \to X_1$$
in a category $\mathcal{C}$. Let $X = \text{“}\varprojlim\text{”}_{i\in\mathbb{N}} X_i$ be the object of pro$-\mathcal{C}$ determined by this sequence, and let $X' = \text{“}\varprojlim\text{”}_{j\in\mathbb{N}} X_{2j}$ be the object of pro$-\mathcal{C}$ determined by taking every other term of the sequence. Show that X and X' are isomorphic as objects of pro$-\mathcal{C}$.

2. Let $\mathcal{C}$ be the category of finite sets. Show that the pro-category pro$-\mathcal{C}$ is equivalent to the category of compact, Hausdorff, totally disconnected topological spaces.

3. Let $\hat{F}_2$ denote the free profinite group on two generators x, y (i.e. for any profinite group G and any elements $a, b \in G$, there is a unique continuous group homomorphism $f\colon \hat{F}_2 \to G$ such that $f(x) = a$ and $f(y) = b$). Let $\Gamma^\bullet \hat{F}_2$ be the descending central series. Show that
$$\bigcap_n \Gamma^n \hat{F}_2 \neq 1$$
is not the trivial subgroup.

4. Let $\mathfrak{u}$ be a finite-dimensional nilpotent Lie algebra (over a characteristic 0 field, as always) of class 3, i.e. $\Gamma^4\mathfrak{u} = 0$. Let $(-) \bullet (-)$ denote the Baker–Campbell–Hausdorff group law on $\mathfrak{u}$. For $u, v \in \mathfrak{u}$ give a formula for the group commutator $u \bullet v \bullet u^{-1} \bullet v^{-1}$ in terms of the Lie bracket. Show that u and v commute for the Baker–Campbell–Hausdorff group law (i.e. $u \bullet v = v \bullet u$) if and only if they commute as elements of the Lie algebra $\mathfrak{u}$ (i.e. $[u, v] = 0$).

5. Let t be the standard coordinate on $\mathbb{G}_a = \mathbb{A}^1_F$, so that $\mathcal{O}(\mathbb{G}_a) = F[t]$. Write down explicitly in terms of t the comultiplication and counit on $\mathcal{O}(\mathbb{G}_a)$. Show that $F[\![\mathbb{G}_a]\!] = \mathcal{O}(\mathbb{G}_a)^* = F[\![N]\!]$ is the power series algebra in one variable N, and describe the coproduct
$$\Delta\colon F[\![\mathbb{G}_a]\!] \to F[\![\mathbb{G}_a]\!] \hat{\otimes}_F F[\![\mathbb{G}_a]\!]$$

on $F[\![\mathbb{G}_a]\!]$. Show that the image of Δ is not contained in the uncompleted tensor product $F[\![\mathbb{G}_a]\!] \otimes_F F[\![\mathbb{G}_a]\!]$ (so $F[\![\mathbb{G}_a]\!]$ is genuinely a complete Hopf algebra, not a Hopf algebra in the usual sense).