Lecture 3: Unipotent and pro-unipotent groups

The profinite étale fundamental group(oid) of a scheme X is a rich and subtle invariant but correspondingly hard to study. For example, if $X = \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0,1,\infty\}$, then we know $\pi_1^{\acute{e}t}(X;x) \simeq \hat{F}_2$ is profinite free on 2 generators. But there is also a natural action of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\pi_1^{\acute{e}t}(X;x)$, and we do not know what this action is in any explicit sense (though there are some things one can say).

The problem is essentially that $\overset{\Pi}{\pi_1^{\acute{e}t}}(X;x)$ is very non-abelian. Specifically let $\Gamma^\bullet\Pi$ denote the descending central series of Π, defined by $\Gamma^1\Pi = \Pi$,

$\Gamma^2\Pi = \overline{[\Pi,\Pi]}$, ← closure of subgroup generated by commutators.

and $\Gamma^{n+1}\Pi = \overline{[\Gamma^n\Pi,\Pi]}$ for $n \geq 1$.

This gives a decreasing sequence of normal subgroups

$$\Pi = \Gamma^1\Pi \trianglerighteq \Gamma^2\Pi \trianglerighteq \dots$$

in which each quotient $\Gamma^n\Pi / \Gamma^{n+1}\Pi$ is <u>abelian</u> and each

$$1 \to \Gamma^n\Pi / \Gamma^{n+1}\Pi \to \Pi / \Gamma^{n+1}\Pi \to \Pi / \Gamma^n\Pi \to 1$$

is a <u>central extension</u>, i.e. $\Gamma^n\Pi / \Gamma^{n+1}\Pi$ lies in the centre of $\Pi / \Gamma^{n+1}\Pi$.

So each quotient $\Pi / \Gamma^{n+1}\Pi$ is nilpotent, so not too far from being abelian.

However, even for $\Pi = \hat{F}_2$, we can have

$$\bigcap_n \Gamma^n\Pi \neq 1 \qquad \text{(see the problems sheets)}$$

so there is more of Π than is captured by the descending central series.

So, we want to replace the profinite étale fundamental group by something which is easier to study. This will be the <u>$\mathbb{Q}_p$-pro-unipotent étale fundamental (groupoid)</u> which is a variant of $\pi_1^{ét}$ which is "$\mathbb{Q}_p$-linear and only mildly non-abelian." In this lecture, we make precise the type of group we are aiming for.

<u>Unipotent groups</u>

Let F be a field of characteristic 0.
A representation V of an affine algebraic group U/F is called <u>unipotent</u> just when there exists a finite U-stable filtration

$$0 = \mathrm{Fil}_{-1} V \subseteq \mathrm{Fil}_0 V \subseteq \mathrm{Fil}_1 V \subseteq \ldots \subseteq \mathrm{Fil}_n V = V$$

such that the U-action on each graded piece $\mathrm{Fil}_i V / \mathrm{Fil}_{i-1} V$ is trivial.

Definition: An affine algebraic group U/F is called *unipotent* just when, equivalently:

1) Every non-zero representation V of U has a non-zero fixed vector: $V^U \neq 0$.
2) Every representation of U is unipotent.
3) U is a closed subgroup of the standard unipotent group

$$Un_m = \left\{ \begin{pmatrix} 1 & * & * & \cdots & * \\ & 1 & * & \cdots & \vdots \\ & & 1 & \ddots & * \\ & & & \ddots & * \\ 0 & & & & 1 \end{pmatrix} \right\} \leq GL_m$$

i.e. ~~strictly~~ upper triangular matrices with 1's on the diagonal.

for some m.

Examples

1. $\mathbb{G}_a = Un_2 = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$ is unipotent.
2. More generally, a finite-dimensional F-vector space V can be viewed as an affine space over F, specifically as the space

$$\mathbb{A}(V) := \operatorname{Spec}(\operatorname{Sym}^\bullet(V^*))$$

Equivalently, $\mathbb{A}(V)$ is the affine scheme representing the functor

$$\begin{aligned} \{F\text{-algebras}\} &\longrightarrow \underline{\mathrm{Set}} \\ \Lambda &\longmapsto \Lambda \otimes_F V. \end{aligned}$$

The additive group law on each $\Lambda \otimes_F V$ determines (via Yoneda) a group law on $\mathbb{A}(V)$, making it into an affine ~~group~~ algebraic group $\mathbb{G}(V)$, called ~~th~~ *vector group*.

Proposition: The vector groups are exactly the abelian unipotent groups. The functor

$$\begin{aligned} \underline{\mathrm{Vec}}_F = \{\text{f.d. vector spaces}\} &\longrightarrow \{\text{vector groups}\} \\ V &\longmapsto \mathbb{G}(V) \end{aligned}$$

is an equivalence. $\boxed{\text{N.B. } \mathbb{G}(V) \simeq \mathbb{G}_a^{\dim(V)}}$

3. For a non-abelian example, consider the Heisenberg group $Un_3 = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ 0 & & 1 \end{pmatrix} \right\}$.

There is a homomorphism

$$\mathrm{Un}_3 \longrightarrow \mathbb{G}_a^2$$
$$\begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \longmapsto (a, b)$$

and a homomorphism

$$\mathbb{G}_a \longrightarrow \mathrm{Un}_3$$
$$c \longmapsto \begin{pmatrix} 1 & 0 & c \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

making Un_3 into a central extension

$$1 \longrightarrow \mathbb{G}_a \longrightarrow \mathrm{Un}_3 \longrightarrow \mathbb{G}_a^2 \longrightarrow 1.$$

Something similar happens in general. Let U be a unipotent group, and

$$U = \Gamma^1 U \trianglerighteq \Gamma^2 U \trianglerighteq \Gamma^3 U \trianglerighteq \dots$$

its descending central series. Then

1. The descending central series is finite, i.e. $\Gamma^n U = 1$ for $n \gg 0$.

2. Each partial quotient

$$V_n := \Gamma^n U / \Gamma^{n+1} U$$

is abelian and unipotent, so a vector group.

3. Each extension

$$\begin{tikzcd}
1 \arrow[r] & \Gamma^n U / \Gamma^{n+1} U \arrow[r] \arrow[d, phantom, "\shortparallel"] & U / \Gamma^{n+1} U \arrow[r] \arrow[d, phantom, "\shortparallel"] & U / \Gamma^n U \arrow[r] \arrow[d, phantom, "\shortparallel"] & 1 \\
& V_n & U_n & U_{n-1} &
\end{tikzcd}$$

is a central extension.

So we've seen (or at least asserted) the following.

<u>Proposition</u>: ("Unipotent groups are iterated central extensions of vector groups")

The category $\underline{\mathrm{Unipt}}_F$ of unipotent groups is the smallest subcategory of affine algebraic groups containing the vector groups and closed under central extensions. $\underline{\mathrm{Unipt}}_F$ is closed under all extensions, subobjects, quotients and all finite limits.

Lie Algebras

The structure of unipotent groups can be understood very explicitly via their Lie algebras.

Definition: Let $\mathfrak{u}$ be a Lie algebra over F. Its descending central series is the sequence of Lie ideals defined by

$$\Gamma^1\mathfrak{u} = \mathfrak{u}, \quad \Gamma^{n+1}\mathfrak{u} = [\Gamma^n\mathfrak{u}, \mathfrak{u}] \quad \text{for } n \geq 1.$$

$\mathfrak{u}$ is called nilpotent just when $\Gamma^n\mathfrak{u} = 0$ for $n >> 0$.

If U is a unipotent group over F, then

$$\Gamma^n \mathrm{Lie}(U) \leq \mathrm{Lie}(\Gamma^n U)$$ (in fact, this is an equality)

so $\mathrm{Lie}(U)$ is nilpotent.

Theorem: The functor $U \longmapsto \mathrm{Lie}(U)$ is an equivalence of categories

$$\{\text{unipotent groups}/F\} \longrightarrow \{\text{finite-dimensional nilpotent Lie algebras}/F\}$$

We won't prove this, but we will at least say what the quasi-inverse functor is. For this, we need the <u>BAKER–CAMPBELL–HAUSDORFF series</u>

$$BCH(x,y) = x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] - \frac{1}{12}[y,[x,y]] + \dots$$

This is the power series in non-commuting variables x, y defined by

$$BCH(x,y) = \log(\exp(x) \cdot \exp(y))$$

where log and exp are the usual power series.

<u>Fact:</u> $BCH(x,y)$ can be written as an infinite $\mathbb{Q}$-linear combination of iterated commutators of x and y, the first few terms of which are as above. (The commutator of two elements a, b in an associative algebra is $[a,b] := ab - ba$.)

If $\mathfrak{u}$ is a finite-dimensional nilpotent Lie algebra over F, we can use the BCH series to define a new binary operation on $\mathfrak{u}$, the <u>BCH product</u>, given by

$$u \bullet v := BCH(u,v) = u + v + \frac{1}{2}[u,v] + \frac{1}{12}[u,[u,v]] - \frac{1}{12}[v,[u,v]] + \dots$$

where the brackets are to be interpreted as the Lie bracket on $\mathfrak{u}$. Note that the sum on the right has only finitely many non-zero terms, by nilpotence of $\mathfrak{u}$, so $BCH(u,v)$ is well-defined.

<u>Lemma</u>: The BCH product makes $\mathfrak{u}$ into a group, with identity $0 \in \mathfrak{u}$ and inverses $u^{-1} = -u$.

<u>Proof</u>: Suppose first that $\mathfrak{u} = \mathcal{H}_m$ is the Lie algebra of strictly upper-triangular $m \times m$ matrices. Then

$$u \bullet v = \log(\exp(x) \cdot \exp(y))$$

where log and exp are interpreted inside the algebra of $m \times m$ matrices. So exp gives an isomorphism

$$(\mathcal{H}_m, \bullet) \xrightarrow{\sim} (Un_m(F), \text{matrix multiplication})$$

so $(\mathcal{H}_m, \bullet)$ is a group. ~~In general, $\mathfrak{u}$ can be~~

A general finite-dimensional nilpotent Lie algebra $\mathfrak{u}$ can be embedded as a Lie subalgebra of $\mathfrak{n}_m$, and this makes $(\mathfrak{u}, \bullet)$ into a subgroup of $(\mathfrak{n}_m, \bullet)$. $\square$

More generally, for any F-algebra Λ, the BCH series defines a group structure on $\Lambda \otimes_F \mathfrak{u}$, functorially in Λ. So this induces a group law on the affine space $\mathbb{A}(\mathfrak{u})$ associated to $\mathfrak{u}$, making it into an affine algebraic group $\mathbb{G}(\mathfrak{u})$. It is not too hard to see that $\mathbb{G}(\mathfrak{u})$ is unipotent (e.g. by embedding $\mathfrak{u}$ inside $\mathfrak{n}_m$). The functor $\mathfrak{u} \longmapsto \mathbb{G}(\mathfrak{u})$ is the desired quasi-inverse to $U \longmapsto \mathrm{Lie}(U)$.

Consequences of the equivalence

1. If U is unipotent, then $U(F)$ is a uniquely divisible group, i.e. for any $u \in U(F)$ and $n \in \mathbb{N}$, there is a unique element $u^{1/n} \in U(F)$ s.t. $(u^{1/n})^n = u$.

Proof: We may suppose that $U = \mathbb{G}(\mathfrak{u})$. Then $u^{1/n} = \frac{1}{n} \cdot u$ is the unique element which works. $\square$

2. If $f : U' \longrightarrow U$ is a homomorphism of unipotent groups, TFAE

i. f is surjective on underlying topological spaces
ii. f is dominant
iii. f is faithfully flat
iv. f is smooth
v. f is surjective on F-points ($f : U'(F) \longrightarrow U(F)$ is surjective)
v'. f is surjective on Λ-points for any F-algebra Λ
vi. f is split as a morphism of F-schemes.

Proof: We may assume $f = \mathbb{G}(g)$ for a homomorphism $g : \mathfrak{u}' \longrightarrow \mathfrak{u}$ of f.d. nilpotent Lie algebras. So, as a morphism of F-schemes, f factors as

$$\mathbb{A}(\mathfrak{u}') \twoheadrightarrow \mathbb{A}(\mathrm{im}(g)) \hookrightarrow \mathbb{A}(\mathfrak{u})$$

affine projection between affine spaces

affine inclusion between affine spaces.

If g is surjective, then f is an affine projection, so i – vi all hold. And if g is not surjective, then i — vi do not hold. $\square$.

<u>Remark</u>: i – iv are equivalent for any homomorphism of connected affine algebraic groups in characteristic 0. But v and vi are special to unipotent groups, e.g. the squaring map

$$[2] : \mathbb{G}_{m,\mathbb{Q}} \longrightarrow \mathbb{G}_{m,\mathbb{Q}}$$

is surjective, but the map on $\mathbb{Q}$-points

$$[2] : \mathbb{Q}^\times \longrightarrow \mathbb{Q}^\times$$

is not.

Hopf algebras

The other way we can study unipotent groups is through their Hopf algebras. If U is an affine algebraic group, there are two natural ways to associate a Hopf algebra to U, namely:

1. the affine ring, or algebra of functions, $\mathcal{O}(U)$
2. the group algebra $F[\![U]\!] := \mathcal{O}(U)^*$

Remark: These are closely analogous to the two ways to associate a Hopf algebra to any finite group G: either the algebra of functions F^G or the group algebra $F[G]$

Some care is required when working with group algebras $F[\![U]\!]$, which are not quite Hopf algebras in the usual sense. The problem comes from the fact that

$(\mathcal{O}(U) \underset{F}{\otimes} \mathcal{O}(U))^* \neq \mathcal{O}(U)^* \underset{F}{\otimes} \mathcal{O}(U)^*$ in general, so the multiplication on $\mathcal{O}(U)$ does not dualise to a comultiplication $F[\![U]\!] \longrightarrow F[\![U]\!] \underset{F}{\otimes} F[\![U]\!]$.

Instead, $F[\![U]\!]$ will be a Hopf algebra in the category of pro-finite dimensional vector spaces.

Definition: The category pro-$\underline{\mathrm{Vec}}_F$ of *pro-finite-dimensional vector spaces* over F is, by definition, the pro-category of the category $\underline{\mathrm{Vec}}_F$ of finite-dimensional vector spaces. So objects of pro-$\underline{\mathrm{Vec}}_F$ are formal cofiltered limits

$$\text{“}\varprojlim_{i\in I}\text{”}\, V_i$$

of finite-dimensional vector spaces. There is a *completed tensor product* $\hat{\otimes}$ (or $\hat{\otimes}_F$) on pro-$\underline{\mathrm{Vec}}_F$:

$$\text{“}\varprojlim_{i\in I}\text{”}\, V_i \;\hat{\otimes}_F\; \text{“}\varprojlim_{j\in J}\text{”}\, W_j := \text{“}\varprojlim_{(i,j)\in I\times J}\text{”}\,(V_i \otimes_F W_j).$$

To understand this category, we use

Proposition: pro-$\underline{\mathrm{Vec}}_F$ is dual to the category $\underline{\mathrm{Mod}}_F$ of all vector spaces, taking $\hat{\otimes}$ to $\otimes$.

Proof: In one direction, if $V = \text{“}\varprojlim_{i\in I}\text{”}\, V_i \in$ pro-$\underline{\mathrm{Vec}}_F$, we define $V^* := \varinjlim_{i\in I} V_i^*$ (colimit taken in $\underline{\mathrm{Mod}}_F$).

Conversely, if $W \in \underline{\mathrm{Mod}}_F$, we define

$W^* := \text{“}\varprojlim_{i\in I}\text{”}\, W_i^*$, where W_i ranges over finite-dimensional subspaces of W. It is easy to check these are quasi-inverse constructions. $\square$

__Corollary:__ pro-$\underline{\mathrm{Vec}}_F \simeq (\underline{\mathrm{Mod}}_F)^{op}$ is an F-linear abelian tensor category, satisfying Grothendieck's axioms AB3+AB4 (existence + exactness of small coproducts) and AB3*+AB4*+AB5* (existence + exactness of small products and cofiltered limits).

⚠ In pro-$\underline{\mathrm{Vec}}_F$, it is the cofiltered limits which are well-behaved, not the filtered colimits. This is the opposite of what happens in $\underline{\mathrm{Mod}}_F$.

We can use this to make sense of what kind of object the group algebra $F[\![u]\!]$ is.

__Definition:__ A __complete Hopf algebra__ is a Hopf algebra object in the tensor category (pro-$\underline{\mathrm{Vec}}_F, \hat{\otimes}, F$). I.e. it is a pro-finite dimensional vector space H with:

- multiplication $\mu : H \hat{\otimes} H \longrightarrow H$
- unit $\eta : F \longrightarrow H$
- comultiplication $\Delta : H \longrightarrow H \hat{\otimes} H$
- counit $\varepsilon : H \longrightarrow F$
- antipode $S : H \longrightarrow H$

satisfying the usual list of axioms for Hopf algebras.

So $F[\![U]\!] = \mathcal{O}(U)^*$ is a cocommutative complete Hopf algebra (with respect to the dualised Hopf algebra operations). The affine algebraic group U being unipotent corresponds to a particularly natural condition on the Hopf algebra side.

<u>Definition:</u> Let H be a complete cocommutative Hopf algebra. Its <u>augmentation ideal</u> $I \trianglelefteq H$ is the kernel of the counit $\varepsilon : H \longrightarrow F$. For $n \geq 1$, we define I^n to be the image of the multiplication map $I^{\hat{\otimes} n} \longrightarrow H$.

We say that H is <u>I-adically complete</u> just when, equivalently,

1. $\bigcap_n I^n = 0$

$\Updownarrow$

2. The map $H \longrightarrow \varprojlim_n (H/I^{n+1})$ is an isomorphism in pro-$\underline{\mathrm{Vec}}_F$.

<u>Proof of equivalence:</u> Since cofiltered limits in pro-$\underline{\mathrm{Vec}}_F$ are exact, we can take the limit of the exact sequences

$$0 \to I^{n+1} \longrightarrow H \longrightarrow H/I^{n+1} \longrightarrow 0$$

to get an exact sequence

$$0 \longrightarrow \bigcap_n I^n \longrightarrow H \longrightarrow \varprojlim_n (H/I^{n+1}) \longrightarrow 0.$$

The result follows. $\square$.

<u>Proposition:</u> U is unipotent if and only if $F[\![U]\!]$ is I-adically complete.

<u>Proof</u> (omitted in lecture): There are $\otimes$-equivalences of $\otimes$-categories

$$\{\text{representations of } U\} \longleftrightarrow \{\text{finite-dimensional } \mathcal{O}(U)\text{-comodules}\}$$
$$\longleftrightarrow \{\text{finite-dimensional } F[\![U]\!]\text{-modules}\}$$

So it suffices to show that a complete cocommutative Hopf algebra H is I-adically complete if and only if every f.-d. representation of H is unipotent, i.e. has a filtration

$$0 = \mathrm{Fil}_{-1}V \subseteq \mathrm{Fil}_0V \subseteq \mathrm{Fil}_1V \subseteq \dots \subseteq \mathrm{Fil}_nV = V$$

by H-submodules such that the H-action on each $\mathrm{Fil}_iV/\mathrm{Fil}_{i-1}V$ factors through $\varepsilon: H \to F$.

In one direction, if H is I-adically complete, then for any f.d. H-module V we can define a filtration on V by

$$I^nV := \mathrm{im}(I^n \hat{\otimes} V \longrightarrow V)$$

Since H is I-adically complete, we have $\bigcap_n I^n V = 0$, and since V is finite-dimensional, we have $I^n V = 0$ for $n \gg 0$. So $I^\bullet V$ is a finite filtration on V, on whose graded pieces I acts by zero. So V is unipotent.

Conversely, suppose that every f.-d. H-module is unipotent. We can write H as a cofiltered limit of finite-dimensional algebras H_i (this is dual to the fact that every coalgebra (in $\underline{\mathrm{Mod}}_F$) is the union of its finite-dimensional subcoalgebras). Each H_i is an H-module in a natural way, so is unipotent, with a filtration

$$0 = \mathrm{Fil}_{-1} H_i \leq \mathrm{Fil}_0 H_i \leq \mathrm{Fil}_1 H_i \leq \ldots \leq \mathrm{Fil}_m H_i = H_i$$

Since I acts trivially on each $\mathrm{Fil}_j H_i / \mathrm{Fil}_{j-1} H_i$, we have $I^n H_i \leq \mathrm{Fil}_{m-n} H_i$ for all n, so $I^n H_i = 0$ for $n \gg 0$. So $\bigcap_n I^n = \varprojlim_{i \in I} \left(\bigcap_n I^n H_i \right) = 0$ and H is I-adically complete. $\square$.

Pro-unipotent groupoids

The final generalisation we need is to -oidify our previous definitions.

Definition: A *groupoid in affine F-schemes* consists of

- a set V of "vertices"
- for $x, y \in V$, an affine F-scheme $\mathcal{U}(x,y)$
- for $x, y, z \in V$, a "composition map"

$$\mathcal{U}(y,z) \times \mathcal{U}(x,y) \longrightarrow \mathcal{U}(x,z)$$

(morphism of affine F-schemes)

- for $x \in V$ an "identity" $1_x \in \mathcal{U}(x,x)(F)$
- for $x, y \in V$, an "inverse map"

$$\mathcal{U}(x,y) \longrightarrow \mathcal{U}(y,x).$$

These should satisfy the usual axioms (associativity, identities, inverses).

For any vertex $x \in V$, $\mathcal{U}(x) := \mathcal{U}(x,x)$ is an affine group-scheme over F. We say that $\mathcal{U}$ is *pro-unipotent* just when $\mathcal{U}(x)$ is ~~unip~~ pro-unipotent for all $x \in V$.