Math 283Z Final Exam

7th–8th May 2023

- You have **24 hours** to complete this exam, from midday on the 7th of May to midday on the 8th of May. I expect the exam to take around 3 hours, but you may use as much or as little of this 24-hour period as you wish. Please continue to take care of yourself during the exam: make sure you are well-rested, and take breaks as and when you need them.
- In the first hour after you receive the exam, you may ask **questions of clarification** about the problems. After this hour, I will not offer any further clarification about the questions on the exam.
- There are four questions on this exam. Your grade for the exam will be based on the **highest three** scores on individual questions. You are not required to attempt all questions.
- The final question is a structured essay question, for which you will summarise and survey some piece of theory from the course. For this question, marks are available for **quality of exposition**, meaning how **clearly** and **precisely** you communicate mathematics.
- You may refer to your lecture notes, the lecture notes available on my website, or any other written sources during the exam. Any result you use in a solution must be **clearly and precisely stated**, and if it comes from another written source, you should include sufficient **bibliographic information** for me to locate the exact source you used. All writing must be exclusively yours.
- Good luck! You've got this.

Problem 1

Let U be a pro-unipotent group over a field F of characteristic 0.

1. Write down the universal property satisfied by the abelianisation U^{ab} of U.
2. Show that if $S \subseteq U(F)$ is a set of F-rational points on U, then the subgroup generated by S is Zariski-dense in U if and only if the subgroup generated by its image in U^{ab} is Zariski-dense. Deduce that U is finitely generated (meaning that there exists a *finite* set S such that the subgroup it generates is Zariski-dense) if and only if U^{ab} is finite-dimensional, i.e. a vector group.
3. Show that if $\phi\colon U_1 \to U_2$ is a homomorphism of pro-unipotent groups such that $\phi^{\mathrm{ab}}\colon U_1^{\mathrm{ab}} \to U_2^{\mathrm{ab}}$ is surjective, then ϕ is surjective. Show moreover that if ϕ^{ab} is an isomorphism and U_2 is free, then ϕ is an isomorphism.

Results from class may be used without proof, provided that they are clearly and precisely stated.

Problem 2

Let F be a characteristic 0 field. Let $\mathcal{T}$ denote the following $\otimes$-category. The objects of $\mathcal{T}$ are finite-dimensional graded vector spaces

$$V = \bigoplus_{i\in\mathbb{Z}} V_i$$

over F, endowed with an endomorphism

$$N = N_V\colon V \to V$$

which is graded of degree -2 ($N(V_i) \subseteq N(V_{i-2})$) with the property that

$$N^i\colon V_i \to V_{-i}$$

is an isomorphism for all $i \geq 0$. Morphisms are defined in the obvious way. The tensor product on $\mathcal{T}$ is given by the usual tensor product $V^1 \otimes V^2$ of graded vector spaces, endowed with the endomorphism $N_{V^1\otimes V^2} = N_{V^1} \otimes 1_{V^2} + 1_{V^1} \otimes N_{V^2}$ where 1_V denotes the identity on V. The tensor unit is $\mathbf{1} = F$, placed in degree 0 for the grading, with $N_{\mathbf{1}} = 0$. You may use without proof the fact that this $\otimes$-category is neutral Tannakian over F, and that the forgetful functor is a fibre functor.

1. Show that for all $n \geq 1$ there is a unique simple object of $\mathcal{T}$ of dimension n (simple means that it has no subobjects in $\mathcal{T}$ other than itself and the zero subobject). Describe this object explicitly. Show moreover that every object of $\mathcal{T}$ decomposes as a direct sum of simple objects.

2. Let V be a representation of SL_2.

   (a) Let $\mathbb{T} \subset \mathrm{SL}_2$ denote the algebraic subgroup consisting of elements of the form
   $$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$
   so that $\mathbb{T}$ is isomorphic to the multiplicative group $\mathbb{G}_m$. Briefly describe (without proof) how the action of $\mathbb{T}$ on V corresponds to a grading on V.

   (b) Let $\mathbb{U} \subset \mathrm{SL}_2$ denote the algebraic subgroup consisting of elements of the form
   $$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix},$$
   so that $\mathbb{U}$ is isomorphic to the additive group $\mathbb{G}_a$. Briefly describe (without proof) how the action of $\mathbb{U}$ on V corresponds to a nilpotent endomorphism N of V.

   (c) Show that the above grading and nilpotent endomorphism make V into an object of $\mathcal{T}$.

3. Prove that the above construction defines a $\otimes$-equivalence from the category of SL_2-representations to $\mathcal{T}$. Conclude that the Tannaka group of $\mathcal{T}$, based at the forgetful functor, is isomorphic to SL_2.

Standard results about the representation theory of SL_2 may be used without proof, provided that they are clearly and precisely stated. You may look up references for the representation theory of SL_2; if you do, you must include inline citations where these results are used.

Problem 3

Let $E/\mathbb{Q}$ be an elliptic curve, and let $Y = E \setminus \{0\}$ be the complement of the identity in E. Let U be the $\mathbb{Q}_p$-pro-unipotent étale fundamental group of Y based at some rational basepoint b. Let

$$U = \mathrm{W}_{-1}U \trianglerighteq \mathrm{W}_{-2}U \trianglerighteq \dots$$

denote the weight filtration on U (which is equal to the descending central series up to reindexing), and adopt the usual notation

$$U_n := \frac{U}{\mathrm{W}_{-n-1}U} \quad \text{and} \quad V_n := \frac{\mathrm{W}_{-n}U}{\mathrm{W}_{-n-1}U}.$$

1. What is the dimension of V_1 and V_2?
2. Show that the commutator pairing in the group U_2 induces an isomorphism

$$\bigwedge^2 V_1 \cong V_2$$

of $G_{\mathbb{Q}}$-representations. Using this, describe the Galois action on V_2. [Hint: use the Weil pairing of the elliptic curve E, recalling that V_1 is the $\mathbb{Q}_p$-linear Tate module of E.]
3. Now assume that E has Mordell–Weil rank 1 and finite Tate–Shafarevich group. Determine the dimensions of the global Bloch–Kato Selmer groups

$$\mathrm{H}^1_f(G_{\mathbb{Q}}, V_1) \quad \text{and} \quad \mathrm{H}^1_f(G_{\mathbb{Q}}, V_2)\,.$$

Determine also the dimensions of the local Bloch–Kato Selmer groups

$$\mathrm{H}^1_f(G_p, V_1) \quad \text{and} \quad \mathrm{H}^1_f(G_p, V_2)\,.$$

4. Under the above assumptions, conclude that $\mathcal{Y}(\mathbb{Z})$ is finite for every integral model $\mathcal{Y}$ of Y. [Use the Chabauty–Kim criterion for the quotient U_2.]

Results from class may be used without proof, provided that they are clearly and precisely stated.

Problem 4

Write an essay on the profinite étale fundamental group, explaining how Grothendieck generalised the notion of fundamental groups of topological spaces to schemes. Your essay should include:

- A definition of the profinite étale fundamental group, including the structure maps.
- A definition of the topology on the étale fundamental group, and a proof that it is profinite. (You do not need to prove that the group operations are continuous.)
- A statement and proof of the comparison theorem with the usual fundamental group of schemes of finite type over $\mathbb{C}$.
- One or more examples where you determine the fundamental group/oid of a scheme.

You may assume that your reader is familiar with the basic language of algebraic geometry, including the definition and basic properties of finite and étale morphisms. Results which were not proved in class (e.g. the Riemann Existence Theorem for coverings) do not need to be proved, but you should clearly and precisely state any results you rely on.